

MA 341 Test 3 Version 1

1. (30 points) Use $\vec{x}' = \begin{bmatrix} 7 & 2 & 0 \\ -6 & 0 & 0 \\ 2 & 1 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ to answer the following:
- Find its complementary solution if its characteristic equation is $(r - 3)^2(r - 4) = 0$
 - Find a particular solution to $\vec{x}' = \begin{bmatrix} 7 & 2 & 0 \\ -6 & 0 & 0 \\ 2 & 1 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ using the **method of undetermined coefficients**. You don't need part a) to do part b).

2. (26 points) Find the particular solution to $\vec{x}' = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ using

the **method of variation of parameters** if $\vec{x}_c = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$

Hint: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3. (16 points) Tank A initially holds 113 L of pure water; tank B initially holds 21 L of a brine solution containing 0.2 kg of dissolved salt. The tanks are connected by pipes. The liquid in tank A flows into tank B at a rate of 10 L/min and from tank B into tank A at a rate of 3 L/min. A solution containing 0.1 kg/L of salt is poured into tank A at a rate of 7 L/min and pure water enters tank B at a rate of 1 L/min. Both tanks are well-mixed. The contents of tank B flow out of a drain at the bottom of tank B at a rate of 8 L/min.

If $x_1(t)$ is the amount of salt in tank A and $x_2(t)$ is the amount of salt in tank B, find the coefficient matrix A , \vec{f} , and $\vec{x}(0)$ so that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \vec{f}$

Do not solve this system!

4. (16 points) Find the general solution to the system $\begin{cases} \frac{dx_1}{dt} = 2x_1 + x_2 \\ \frac{dx_2}{dt} = -13x_1 - 2x_2 \end{cases}$

You may write your final answer in the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Hint: If $r = \alpha + \beta i$ with $\mathbf{u} = \mathbf{a} + \mathbf{b}i$ then two linearly independent solutions are $e^{\alpha t}(\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b})$ and $e^{\alpha t}(\cos(\beta t)\mathbf{b} + \sin(\beta t)\mathbf{a})$

5. (12 points) Find the inverse of $A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 3 & -1 & 3 \\ 0 & 2 & 1 \end{bmatrix}$ using row operations

MA 341 Test 3 Version 1 Solution

1. (30 points)

a) $(A-3I)\vec{u} = \vec{0}$

$$\begin{bmatrix} 4 & 2 & 0 \\ -6 & -3 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} = \vec{0}$$

$$2u_a + u_b = 0$$

$$u_b = -2u_a$$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$(A-4I)\vec{u}_3 = \vec{0}$

$$\begin{bmatrix} 3 & 2 & 0 \\ -6 & -4 & 0 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} = \vec{0}$$

$$3u_a + 2u_b = 0 \rightarrow u_a = -\frac{2}{3}u_b$$

$$2u_a + u_b - u_c = 0$$

$$-4 + 3 - u_c = 0$$

$$u_b = 3$$

$$u_a = -2$$

$$u_c = -1$$

$$\vec{u}_3 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

$$\vec{x}_c = c_1 e^{3t} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

$$b) \vec{x}_p = \vec{a}$$

$$\vec{x}_p' = \vec{0}$$

$$\vec{0} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}' = \begin{bmatrix} 7 & 2 & 0 \\ -6 & 0 & 0 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7a_1 + 2a_2 \\ -6a_1 \\ 2a_1 + a_2 + 3a_3 \end{bmatrix} \quad a_1 = 0 \quad a_2 = -3$$

$$0 = 2a_1 - 3 + 3a_3 \quad a_3 = 1$$

$$\vec{x}_p = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

2. (26 points)

$$\underline{X} = \begin{bmatrix} 1 & 3e^{2t} \\ 1 & e^{2t} \end{bmatrix}$$

$$\begin{aligned} \underline{X}^{-1} &= \frac{1}{e^{2t} - 3e^{2t}} \begin{bmatrix} e^{2t} & -3e^{2t} \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2}e^{-2t} & -\frac{1}{2}e^{-2t} \end{bmatrix} \end{aligned}$$

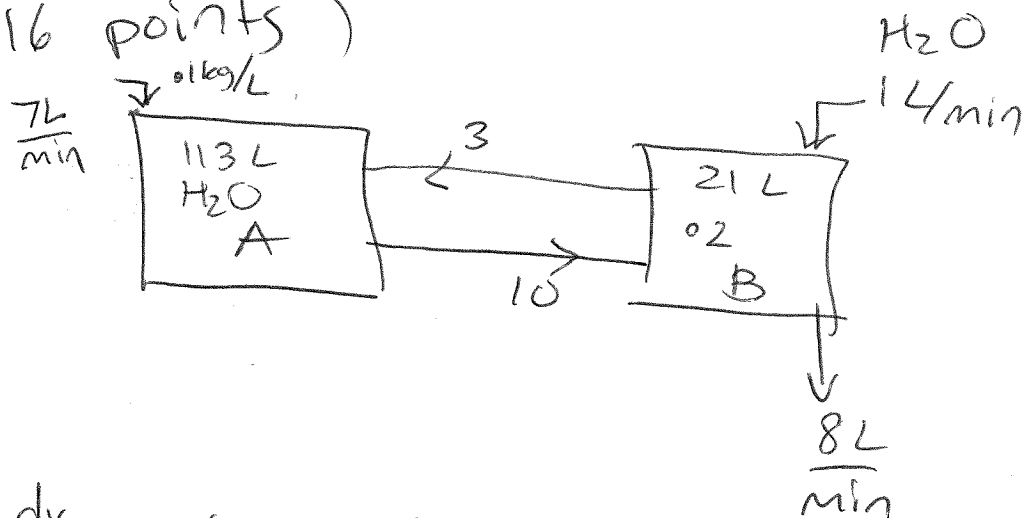
$$\vec{x}_p = \underline{X} \int \underline{X}^{-1} \vec{f} dt$$

$$= \underline{X} \int \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2}e^{-2t} & -\frac{1}{2}e^{-2t} \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix} dt$$

$$= \underline{X} \int \begin{bmatrix} -3 & -3 \\ 3e^{-2t} + e^{-2t} \end{bmatrix} dt = \underline{X} \int \begin{bmatrix} -6 \\ 4e^{-2t} \end{bmatrix} dt$$

$$= \begin{bmatrix} 1 & 3e^{2t} \\ 1 & e^{2t} \end{bmatrix} \begin{bmatrix} -6t - 6 \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} -6t - 6 \\ -6t - 2 \end{bmatrix}$$

3. (16 points)



$$\frac{dx_1}{dt} = 7(0.1) + 3 \frac{x_2}{21} - 10 \frac{x_1}{113}$$

$$\frac{dx_2}{dt} = 1(0) + 10 \frac{x_1}{113} - 11 \frac{x_2}{21}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -\frac{10}{113} & \frac{3}{21} \\ \frac{10}{113} & -\frac{11}{21} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.7 \\ 0 \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}$$

4. (16 points)

$$A = \begin{bmatrix} 2 & 1 \\ -13 & -2 \end{bmatrix}$$

$$|A - rI| = \begin{vmatrix} 2-r & 1 \\ -13 & -2-r \end{vmatrix}$$

$$= (2-r)(-2-r) + 13$$

$$= r^2 - 4 + 13 = r^2 + 9 = 0$$

$$r = \pm 3i$$

$$(2-3i)u_a + u_b = 0$$

$$u_b = -(2-3i)u_a$$

$$(A - 3iI)\vec{u} = \vec{0}$$

$$\begin{bmatrix} 2-3i & 1 \\ -13 & -2-3i \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix} = \vec{0}$$

$$\vec{u} = \begin{bmatrix} 1 \\ -2+3i \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} i$$

$$\vec{x} = C_1 \left(\cos 3t \begin{bmatrix} 1 \\ -2 \end{bmatrix} - \sin 3t \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) + C_2 \left(\cos 3t \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \sin 3t \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right)$$

5. (12 points)

$$\left[\begin{array}{ccc|ccc} \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & 0 \\ 3 & -1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 0 & 0 \\ 3 & -1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$R_2 - 3R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & -6 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 6 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$R_3 - 2R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 6 & -1 & 0 \\ 0 & 0 & 1 & -12 & 2 & 1 \end{array} \right] \xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -2 & -1 \\ 0 & 1 & 0 & 6 & -1 & 0 \\ 0 & 0 & 1 & -12 & 2 & 1 \end{array} \right]$$

A^{-1}