MA 341 Test 3 Version 2

1. (30 points) Use $\vec{x}' = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & -2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 12 \\ 1 \\ -1 \end{bmatrix}$ to answer the following:
   a) Find its complementary solution if its characteristic equation is $(r-4)^2(r+1) = 0$
   b) Find a particular solution to $\vec{x}' = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & -2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 12 \\ 1 \\ -1 \end{bmatrix}$ using the **method of undetermined coefficients**. You don't need part a) to do part b).

2. (26 points) Find the particular solution to $\vec{x}' = \begin{bmatrix} 4 & -4 \\ 1 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} 10 \\ 1 \end{bmatrix}$ using

   the **method of variation of parameters** if $\vec{x}_c = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t}$

   Hint: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3. (16 points) Tank A initially holds 60 L of pure water; tank B initially holds 83 L of a brine solution containing 0.15 kg of dissolved salt. The tanks are connected by pipes. The liquid in tank A flows into tank B at a rate of 8 L/min and from tank B into tank A at a rate of 2 L/min. A solution containing 0.1 kg/L of salt is poured into tank A at a rate of 6 L/min and pure water enters tank B at a rate of 1 L/min. Both tanks are well -mixed. The contents of tank B flow out of a drain at the bottom of tank B at a rate of 7 L/min.

   If $x_1(t)$ is the amount of salt in tank A and $x_2(t)$ is the amount of salt in tank B, find the coefficient matrix **A**, $\vec{\mathbf{f}}$, and $\vec{x}(0)$ so that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \vec{\mathbf{f}}$

   Do not solve this system!

4. (16 points) Find the general solution to the system $\begin{cases} \frac{dx_1}{dt} = 2x_1 + x_2 \\ \frac{dx_2}{dt} = -8x_1 - 2x_2 \end{cases}$

   You may write your final answer in the form $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

   Hint: If r=$\alpha + \beta i$ with **u**=**a** + **b**i then two linearly independent solutions are $e^{\alpha t}(cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b})$ and $e^{\alpha t}(cos(\beta t)\mathbf{b} + \sin(\beta t)\mathbf{a})$

5. (12 points) Find the inverse of A = $\begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ 2 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ using row operations

MA 341 T3 V2 Solutions

1. (30 points)

a) $(A-4I)\vec{u}=\vec{0}$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -2 & -3 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}$$

$$\vec{u_1}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$2u_b + 3u_c = 0$

$u_a = -\frac{3}{2}u_c$

$$\vec{u_2}=\begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$$

$$(A+1I)\begin{bmatrix} 5 & 2 & 3 \\ 0 & 3 & -3 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}$$

$u_b = u_c = 1$

$u_a = -1$

$$\vec{u_3}=\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{X_c} = c_1 e^{4t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

b) $\vec{x_p}=\vec{a}$ $\quad \vec{x_p}'=\vec{0}$

$$\vec{0} = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 12 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{0} = \begin{bmatrix} 4a_1 + 2a_2 + 3a_3 \\ 2a_2 - 3a_3 \\ -2a_2 + a_3 \end{bmatrix} + \begin{bmatrix} 12 \\ 1 \\ -1 \end{bmatrix} \Big\} \; 0 = -2a_3 + 0$$

$a_3 = 0$

$a_2 = 0$

$a_1 = -3$

$$\vec{x_p} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

2. (26 points)

$$\mathbf{X} = \begin{bmatrix} 1 & 4e^{3t} \\ 1 & e^{3t} \end{bmatrix} \qquad \mathbf{X}^{-1} = \frac{1}{e^{3t}-4e^{3t}} \begin{bmatrix} e^{3t} & -4e^{3t} \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & \frac{4}{3} \\ 1/3\, e^{-3t} & -\frac{1}{3} e^{-3t} \end{bmatrix}$$

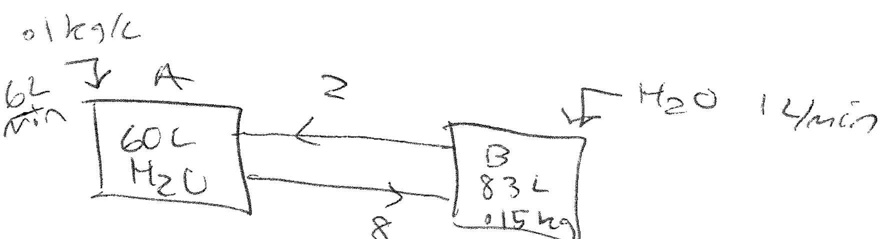
$$\vec{x}_p = \mathbf{X} \int \begin{bmatrix} -1/3 & 4/3 \\ 1/3\, e^{-3t} & -\frac{1}{3e^{-3t}} \end{bmatrix} \begin{bmatrix} 10 \\ 1 \end{bmatrix} dt$$

$$= \mathbf{X} \int \begin{bmatrix} -10/3 + 4/3 \\ 10/3\, e^{-3t} - 1/3\, e^{-3t} \end{bmatrix} dt = \mathbf{X} \int \begin{bmatrix} -6/3 \\ \frac{9}{3} e^{-3t} \end{bmatrix} dt$$

$$= \mathbf{X} \int \begin{bmatrix} -2 \\ 3e^{-3t} \end{bmatrix} dt$$

$$= \begin{bmatrix} 1 & 4e^{3t} \\ 1 & e^{3t} \end{bmatrix} \begin{bmatrix} -2t \\ -e^{-3t} \end{bmatrix} = \begin{bmatrix} -2t-4 \\ -2t-1 \end{bmatrix}_{2\times 1}$$

3. (16 points)



$$\frac{dx_1}{dt} = 6(.1) + 2\,\frac{x_2}{83} - 8\,\frac{x_1}{60}$$

$$\frac{dx_2}{dt} = 1(0) + 8\,\frac{x_1}{60} - 9\,\frac{x_2}{83}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -8/60 & 2/83 \\ 8/60 & -9/83 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} .6 \\ 0 \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} 0 \\ .15 \end{bmatrix}$$

4. (16 points)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 2 & 1 \\ -8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$|A - rI| = \begin{vmatrix} 2-r & 1 \\ -8 & -2-r \end{vmatrix} = r^2 - 4 + 8$$

$$r^2 + 4 = 0$$

$$r^2 = -4$$

$$r = \pm 2i$$

$$(A - 2i)\vec{u} = \vec{0}$$

$$\begin{bmatrix} 2-2i & 1 \\ -8 & -2-2i \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix} = \vec{0}$$

$$(2-2i)u_a + u_b = 0 \qquad u_b = -(2-2i)u_a$$

$$\vec{u} = \begin{bmatrix} 1 \\ -2+2i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{\vec{a}} + \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{\vec{b}} i$$

$$\boxed{\vec{x} = c_1\left(\cos 2t \begin{bmatrix} 1 \\ -2 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) + c_2\left(\cos 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)}$$

5. (12 points)

$$\left[\begin{array}{ccc|ccc} 1/3 & 0 & 1/3 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{3R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$R_2-2R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & 0 & 0 \\ 0 & -1 & 0 & -6 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 6 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$R_3-2R_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 6 & -1 & 0 \\ 0 & 0 & 1 & -12 & 2 & 1 \end{array}\right]$$

$$R_1-R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 15 & -2 & -1 \\ 0 & 1 & 0 & 6 & -1 & 0 \\ 0 & 0 & 1 & -12 & 2 & 1 \end{array}\right]$$

A^{-1}