MA 341 Test 3 Version 2

1. (18 points) Find the general solution $\vec{x}' = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix} \vec{x}$

2. (24 points) Find the particular solution to $\vec{x}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{2t}$ using

   the **method of variation of parameters** if $\vec{x}_c = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

   Hint: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3. (16 points) Tank A initially holds 300 L of pure water; tank B initially holds 100 L of a brine solution containing 2 kg of dissolved salt. The tanks are connected by pipes. The liquid in tank A flows into tank B at a rate of 7 L/min and from tank B into tank A at a rate of 3 L/min. A solution containing 0.1 kg/L of salt is poured into tank A at a rate of 6 L/min and pure water enters tank B at a rate of 1 L/min. Both tanks are well -mixed. The contents of tank A flow out of a drain at the bottom of tank A at a rate of 2 L/min while the contents of tank B flow out of a drain at the bottom of tank B at a rate of 5 L/min.

   If $x_1(t)$ is the amount of salt in tank A and $x_2(t)$ is the amount of salt in tank B, find the coefficient matrix **A**, $\vec{\mathbf{f}}$, and $\vec{x}\,(0)$ so that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \vec{\mathbf{f}}$

   Do not solve this system!

4. (30 points) Use $\vec{x}' = \begin{bmatrix} 6 & 5 \\ -2 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} -30t \\ 10t \end{bmatrix}$ to answer the following:
   a) Find its complementary solution
      Hint: If $r = \alpha + \beta i$ with $\mathbf{u} = \mathbf{a} + \mathbf{b}i$ then two linearly independent solutions are
      $e^{\alpha t}(\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b})$ and $e^{\alpha t}(\cos(\beta t)\mathbf{b} + \sin(\beta t)\mathbf{a})$
   b) Find its particular solution using the **method of undetermined coefficients**. You don't need part a) to do part b).

5. (12 points) Find the inverse of $A = \begin{bmatrix} 4 & 1 & 4 \\ 3 & 1 & 3 \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$ using row operations

MA 341 T3 Solutions

1. (18 points)

$$\begin{vmatrix} 4-r & 2 & 3 \\ 0 & 2-r & -3 \\ 0 & 0 & 4-r \end{vmatrix} = (4-r)^2(2-r) = 0$$

$r_1 = r_2 = 4 \quad r_3 = 2$

$(A-4I)\vec{u} = \vec{0}$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}$$

$\vec{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad 2u_b + 3u_c = 0$

$u_b = -\frac{3}{2}u_c$

$\vec{u_2} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$

$(A-2I)\vec{u_3} = \vec{0}$

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 0 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} = \vec{0}$$

$u_c = 0$

$2u_a + 2u_b = 0$ $\quad u_a = -u_b$ $\quad \vec{u_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{x} = c_1 e^{4t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

2. (24 pts)

$$\Phi = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}$$

$$\Phi^{-1} = \frac{1}{2e^{3t} - e^{3t}} \begin{bmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}$$

$$\vec{X}_p = \Phi \int \Phi^{-1} \vec{f} \, dt$$

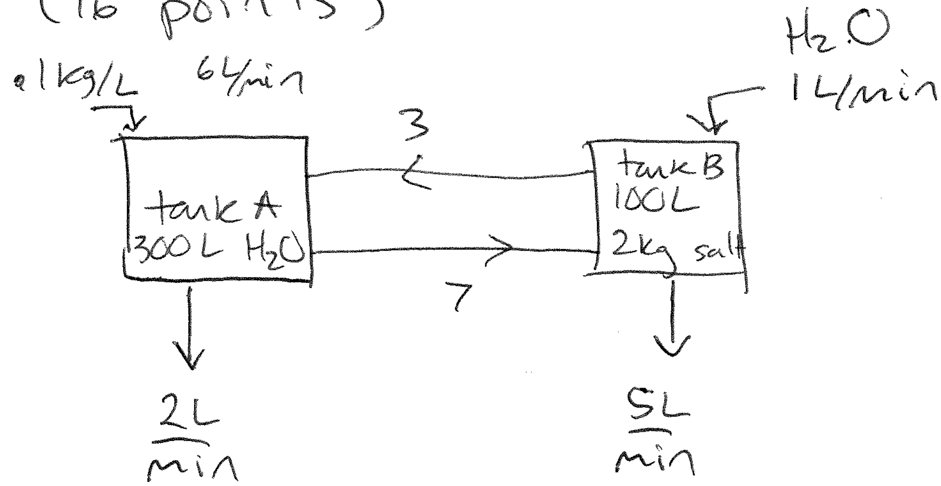
$$= \Phi \int \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix} \begin{bmatrix} 5e^{2t} \\ e^{2t} \end{bmatrix} dt$$

$$= \Phi \int \begin{bmatrix} 5e^t - e^t \\ -5 + 2 \end{bmatrix} dt$$

$$= \Phi \int \begin{bmatrix} 4e^t \\ -3 \end{bmatrix} dt$$

$$= \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} \begin{bmatrix} 4e^t \\ -3t \end{bmatrix} = \boxed{\begin{bmatrix} 8e^{2t} - 3te^{2t} \\ 4e^{2t} - 3te^{2t} \end{bmatrix}}$$

3. (16 points)



$$\frac{dx_1}{dt} = 6(.1) + 3\left(\frac{x_2}{100}\right) - 9\left(\frac{x_1}{300}\right)$$

$$\frac{dx_2}{dt} = 1(0) + 7\left(\frac{x_1}{300}\right) - 8\left(\frac{x_2}{100}\right)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} \frac{-9}{300} & \frac{3}{100} \\ \frac{7}{300} & \frac{-8}{100} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} .6 \\ 0 \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

4. (30 points)

a) $\begin{vmatrix} 6-r & 5 \\ -2 & -r \end{vmatrix} = (6-r)(-r) + 10$

$r^2 - 6r + 10 = 0$

$r = \dfrac{6 \pm \sqrt{36-40}}{2} = \dfrac{6 \pm 2i}{2} = 3 \pm i$

$(A - (3+i)I)\vec{u} = \vec{0}$

$\begin{bmatrix} 3-i & 5 \\ -2 & -3-i \end{bmatrix}\begin{bmatrix} u_a \\ u_b \end{bmatrix} = \vec{0}$

$-2u_a + (-3-i)u_b = 0$

$u_a = \dfrac{(3+i)u_b}{-2}$

$\vec{u} = \begin{bmatrix} 3+i \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$

$\vec{x} = c_1 e^{3t}\left(\cos t \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$+ c_2 e^{3t}\left(\cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)$

4 b)

$$\vec{x_p} = \vec{a}t + \vec{b}$$

$$\vec{x_p}' = \vec{a}$$

$$\vec{a} = \begin{bmatrix} 6 & 5 \\ -2 & 0 \end{bmatrix}(\vec{a}t + \vec{b}) + \begin{bmatrix} -30 \\ 10 \end{bmatrix} t$$

$$\vec{a} = \underbrace{\begin{bmatrix} 6 & 5 \\ -2 & 0 \end{bmatrix}\vec{a}t} + \begin{bmatrix} 6 & 5 \\ -2 & 0 \end{bmatrix}\vec{b} + \underbrace{\begin{bmatrix} -30 \\ 10 \end{bmatrix} t}$$

$$\vec{0} = \begin{bmatrix} 6 & 5 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} -30 \\ 10 \end{bmatrix}$$

$$\vec{0} = \begin{bmatrix} 6a_1 + 5a_2 \\ -2a_1 \end{bmatrix} + \begin{bmatrix} -30 \\ 10 \end{bmatrix}$$

$a_1 = 5$ $\quad$ $0 = 6(5) + 5a_2 - 30$

$a_2 = 0$

$$\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 6b_1 + 5b_2 \\ -2b_1 \end{bmatrix}$$

$b_1 = 0 \quad b_2 = 1$

$$\vec{x_p} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5t \\ 1 \end{bmatrix}$$

5. (12 points)

$$\left[\begin{array}{ccc|ccc} 4 & 1 & 4 & 1 & 0 & 0 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ 0 & -1 & \frac{1}{2} & 0 & 0 & 1 \end{array}\right]$$

$$R_2 - 3R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -3 & 4 & 0 \\ 0 & -1 & \frac{1}{2} & 0 & 0 & 1 \end{array}\right]$$

$$R_3 + R_2 \rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -3 & 4 & 0 \\ 0 & 0 & \frac{1}{2} & -3 & 4 & 1 \end{array}\right]$$

$$2R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -3 & 4 & 0 \\ 0 & 0 & 1 & -6 & 8 & 2 \end{array}\right]$$

$$R_1 - R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -9 & -2 \\ 0 & 1 & 0 & -3 & 4 & 0 \\ 0 & 0 & 1 & -6 & 8 & 2 \end{array}\right]$$

A^{-1}