**MA 241-050 Test 4:** Be sure to show all of your work and **specify every test you use as well as the requirements for each test** as we have done in class.

1. (35 points) Determine if the following series converge or diverge. **Find the sum of convergent series.** Justify your answers thoroughly as we have done in class.

   a) $\sum_{n=1}^{\infty} \frac{1}{n+2} - \frac{1}{n}$ **Include the first three partial sums with your answer**

   b) $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

   c) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{2} + \cdots$

   d) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+5}\right)$

2. (12 points) Use the Integral Test to determine the convergence or divergence of $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$. Briefly mention the two conditions we need to have before we can apply the Integral Test.

3. (22 points) Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{(x-4)^n}{3^n\, n^5}$

4. (16 points) a) Find a power series representation for $f(x)=\frac{1}{8+x^3}$ and determine its radius of convergence. Fully simplify your series as we have done in class.

   b) Use your result from part a) to find $\int_0^1 \frac{1}{8+x^3}\, dx$

   c) What is the maximum error $|R_n|$ using $S_2$ to approximate the sum of the series from part b)? Hint: $|R_n| \leq C_{n+1}$

5. (15 points) Determine if the following series are absolutely convergent, conditionally convergent, or divergent.

   a) $\sum_{n=1}^{\infty} \frac{(-6)^n}{(2n)!}$

   b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+n}}$

# C2 T(2+2) Solutions

1. (35 points)

a) $S_1 = \frac{1}{3} - 1$

$S_2 = \frac{1}{3} - 1 + \frac{1}{4} - \frac{1}{2}$ Telescoping

$S_3 = \cancel{\frac{1}{3}} - 1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} - \cancel{\frac{1}{3}}$

$S_4 = -1 + \cancel{\frac{1}{4}} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \cancel{\frac{1}{4}}$

$S_n = -1 - \frac{1}{2} + \frac{1}{n+1} + \frac{1}{n+2}$

$\lim_{n\to\infty} S_n = \boxed{\frac{-3}{2}}$ converges to

b) $a + ar + ar^2 + \cdots$

$a = 5 \qquad -\frac{10}{3} = 5r \qquad r = -\frac{2}{3}$

$ar^2 = 5\left(-\frac{2}{3}\right)^2 = 5\left(\frac{4}{9}\right) = \frac{20}{9}$ ✓

Geometric series

$|-2/3| < 1$ converges to $\frac{a}{1-r} = \frac{5}{1-\left(-\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = \boxed{3}$

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p-series $p = 1/2 < 1$ diverges

d) $\lim_{n\to\infty} \ln\left(\frac{n}{2n+5}\right) = \ln\left(\frac{1}{2}\right) \neq 0$ diverges

Divergence test

2. (12 points) decreasing, positive ✓

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \tan^{-1}x \Big|_0^t = \lim_{t\to\infty} \tan^{-1}t - \tan^{-1}0 = \frac{\pi}{2}$$

converges

$\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ conv Integral test

3. (22 points)

Ratio test $\lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{3^{n+1}(n+1)^5} \cdot \frac{3^n n^5}{(x-4)^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{(x-4)\,n^5}{3\,(n+1)^5} \right| = \left| \frac{x-4}{3} \right| < 1 \qquad |x-4| < 3$$

$\boxed{R=3}$

Endpts: $x = a-R = 4-3 = 1$ $\qquad x = a+R = 4+3 = 7$

$x=1$: $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n n^5} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ AST

$C_n = \frac{1}{n^5} \geq C_{n+1} = \frac{1}{(n+1)^5}$

dec ✓

$\lim_{n\to\infty} \frac{1}{n^5} = 0$ ✓ conv.

$x=7$ $\sum_{n=1}^{\infty} \frac{3^n}{3^n n^5} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ p-series $p=5>1$ conv

Interval of C: $[1,7]$

5. (15 points)

a) Ratio test

$$\lim_{n\to\infty}\left|\frac{6^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{6^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{6^n 6\,(2n)!}{(2n+2)(2n+1)(2n)!\,6^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{6}{(2n+2)(2n+1)}\right| = 0 < 1$$

Absolutely convergent

b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+n}}$

LCT: $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n^2+n}}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}} = 1 > 0$ finite

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ Harmonic series diverges $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+n}}$ diverges by LCT

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^2+n}}$ AST $\displaystyle C_n=\frac{1}{\sqrt{n^2+n}} \geq C_{n+1}=\frac{1}{\sqrt{(n+1)^2+(n+1)}}$

dec ✓

$\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n^2+n}}=0$ ✓

<u>conditionally convergent</u>

4. (16 points)

a) $$\frac{1}{8(1+\frac{x^3}{8})} = \frac{1}{8(1-(-\frac{x^3}{8}))} = \frac{1}{8}\sum_{n=0}^{\infty}\left(-\frac{x^3}{8}\right)^n$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{8^{n+1}}}$$

$$\left|-\frac{x^3}{8}\right| < 1$$

$$|x^3| < 8$$

$$|x| < 2$$

$$\boxed{R=2}$$

b) $$\int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{8^{n+1}}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+1}}{8^{n+1}(3n+1)}\Big|_0^1$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{8^{n+1}(3n+1)}$$

c) $$\underset{n=0}{\frac{1}{8(1)}} - \underset{n=1}{\frac{1}{8^2(4)}} + \underset{n=2}{\frac{1}{8^3(7)}} - \underset{n=3}{\frac{1}{8^4(10)}}$$

$$|\text{error}| \le \frac{1}{8^4(10)}$$