Name:___________________________ MA 341 Test 2 Version 1

1. (16 points) Use **the method of undetermined coefficients** to solve the Initial Value Problem (IVP): $y'' + 2y' = -16\sin(2t)$; y(0)=0, $y'(0)$=0

2. (12 points) Use **the method of variation of parameters** to find a particular solution to:
   $y'' - 3y' + 2y = 7e^{2t}$

3. (12 points) A 2 kg mass attached to the end of a hanging spring stretches the spring 2 m upon coming to rest at equilibrium. Its damping constant is 8 Ns/m. At t=0, an external force of 30 N is applied to the system. Hint: If it is needed, use g= 10 m/s^2 to answer the following:
   a) If the mass is compressed 2 m from its equilibrium position and released. Formulate the IVP that describes this system (Do not solve)
   b) Find its steady-state solution

4. (10 points) Use the definition of the Laplace transform to find the Laplace transform of $f(t)=9$ state its domain.

Use the table below to answer the following:

$L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$	$L\{\cos(bt)\} = \frac{s}{s^2+b^2}$	$L\{e^{at}\} = \frac{1}{s-a}$
$L\{y'\} = sL\{y\} - y(0)$	$L\{\sin(bt)\} = \frac{b}{s^2+b^2}$	$L\{t^n\} = \frac{n!}{s^{n+1}}$
$L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$	$L\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$	$L\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$
$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$	$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	$L\{t^n f(t)\} = (-1)^n \frac{d^n(F(s))}{ds^n}$
	$L\{1\} = \frac{1}{s}$	$L\{u(t-a)\} = \frac{e^{-as}}{s}$

5. (12 points) Express the given function using unit step functions and find its Laplace transform: $f(t) = \begin{cases} 8, & t < 1 \\ 0, & 1 \le t \le 3 \\ 6e^{2t}, & 3 \le t \end{cases}$

Use the table below to answer the following:

$L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$	$L\{\cos(bt)\} = \frac{s}{s^2+b^2}$	$L\{e^{at}\} = \frac{1}{s-a}$
$L\{y'\} = sL\{y\} - y(0)$	$L\{\sin(bt)\} = \frac{b}{s^2+b^2}$	$L\{t^n\} = \frac{n!}{s^{n+1}}$
$L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$	$L\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$	$L\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$
$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$	$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	$L\{t^n f(t)\} = (-1)^n \frac{d^n(F(s))}{ds^n}$
	$L\{1\} = \frac{1}{s}$	$L\{u(t-a)\} = \frac{e^{-as}}{s}$

6. (7 points) Find the inverse Laplace of the following: $\frac{e^{-7s}}{s^2}$

$L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$	$L\{\cos(bt)\} = \frac{s}{s^2+b^2}$	$L\{e^{at}\} = \frac{1}{s-a}$
$L\{y'\} = sL\{y\} - y(0)$	$L\{\sin(bt)\} = \frac{b}{s^2+b^2}$	$L\{t^n\} = \frac{n!}{s^{n+1}}$
$L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$	$L\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$	$L\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$
$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$	$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	$L\{t^n f(t)\} = (-1)^n \frac{d^n(F(s))}{ds^n}$
	$L\{1\} = \frac{1}{s}$	$L\{u(t-a)\} = \frac{e^{-as}}{s}$

7. (15 points) Find the inverse Laplace of the following: $\frac{20+4s^2}{s(s^2-2s+10)}$

$L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$	$L\{\cos(bt)\} = \frac{s}{s^2+b^2}$	$L\{e^{at}\} = \frac{1}{s-a}$
$L\{y'\} = sL\{y\} - y(0)$	$L\{\sin(bt)\} = \frac{b}{s^2+b^2}$	$L\{t^n\} = \frac{n!}{s^{n+1}}$
$L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$	$L\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$	$L\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$
$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$	$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	$L\{t^n f(t)\} = (-1)^n \frac{d^n(F(s))}{ds^n}$
	$L\{1\} = \frac{1}{s}$	$L\{u(t-a)\} = \frac{e^{-as}}{s}$

8. (16 points) Use the method of Laplace transforms to solve the Initial Value Problem:
   $y'' - 2y' + y = 6$; $y(0)=0$, $y'(0) = 1$

MA 341 T2 Solutions

1. (16 points)

$r^2 + 2r = 0$

$r(r+2) = 0$

$y_c = C_1 + C_2 e^{-2t}$

$y_p = A\cos 2t + B\sin 2t$

$y_p' = -2A\sin 2t + 2B\cos 2t$

$y_p'' = -4A\cos 2t - 4B\sin 2t$

$y'' + 2y' = -16\sin 2t$

$-4A\cos 2t - 4B\sin 2t + 2[-2A\sin 2t + 2B\cos 2t] = -16\sin 2t$

$-4A + 4B = 0 \qquad A = B$

$-4B - 4A = -16$

$-8B = -16 \qquad B = 2$

$y = C_1 + C_2 e^{-2t} + 2\cos 2t + 2\sin 2t$

$y(0) = 0 = C_1 + C_2 + 2$

$y' = 0 - 2C_2 e^{-2t} - 4\sin 2t + 4\cos 2t$

$y'(0) = 0 = -2C_2 + 4 \qquad C_2 = 2 \qquad C_1 = -4$

$$\boxed{y = -4 + 2e^{-2t} + 2\cos 2t + 2\sin 2t}$$

2. (12 points)

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$y_c = c_1 e^{2t} + c_2 e^t$

$y_p = v_1 e^{2t} + v_2 e^t$ $\qquad v_1' y_1 + y_2' y_2 = 0$

$-\left(v_1' e^{2t} + v_2' e^t = 0\right)$

$v_1' 2e^{2t} + v_2' e^t = 7e^{2t}$

$v_1' e^{2t} = 7e^{2t}$

$v_1' = 7$

$v_1 = 7t$

$v_2' = -v_1' \frac{e^{2t}}{e^t} = -v_1' e^t = -7e^t$

$v_2 = -7e^t$

$y_p = 7te^{2t} - 7e^{2t}$

3. (12 points) $my'' + by' + ky = F_{ext}$

a) $F = ky$

$2(10) = k2$ $\qquad k = 10$

$2y'' + 8y' + 10y = 30$
$y(0) = -2, \quad y'(0) = 0$

b) $y_p = A$ $\qquad y_p' = 0$ $\qquad 10A = 30 \qquad A = 3$

$y_p = 3$

4. (10 points)

$$\mathcal{L}\{9\} = \int_0^\infty e^{-st}\, 9\, dt$$

$$= \lim_{n\to\infty} \int_0^n 9e^{-st}\, dt$$

$$= \lim_{n\to\infty} -\frac{9}{s} e^{-st} \Big|_0^n = \lim_{n\to\infty} -\frac{9}{s}\left(e^{-sn} - e^{0}\right) = \boxed{\frac{9}{s}}$$

if $\boxed{s > 0}$

5. (12 points)

$$f(t) = 8 - 8u(t-1) + 6e^{2t}u(t-3)$$

$$\mathcal{L}\{f\} = \frac{8}{s} - \frac{8e^{-s}}{s} + e^{-3s}\mathcal{L}\{6e^{2(t+3)}\}$$

$$= \frac{8}{s} - \frac{8}{s}e^{-s} + e^{-3s}\left[\frac{6e^{6}}{s-2}\right]$$

6. (7 points)

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}e^{-7s}\right\} = (t-7)u(t-7)$$

$$F(s) = \frac{1}{s^2} \quad f(t) = t$$

7. (15 points)

$$\frac{A}{s} + \frac{B(s-1)+3C}{(s-1)^2+3^2}$$

$s^2-2s+10 = (s-\alpha)^2+\beta^2$

$s^2-2\alpha s+\alpha^2+\beta^2$

$\alpha=1 \quad \beta=3$

$A(s^2-2s+10) + (B(s-1)+3C)s = 20+4s^2$

$As^2 - 2As + 10A + Bs^2 - Bs + 3Cs = 20+4s^2$

$A+B=4$

$-2A-B+3C=0$

$10A=20 \quad A=2 \quad B=2$

$-4-2+3C=0$

$C=2$

$2 + 2e^t\cos 3t + 2e^t\sin 3t$

8. (16 points)

$s^2L - sy(0) - y'(0) - 2[sL - y(0)] + L = \frac{6}{s}$

$(s^2-2s+1)L = \frac{6}{s} + 1$

$$L = \frac{\frac{6}{s}+1}{(s-1)^2} \cdot \frac{s}{s} = \frac{6+s}{s(s-1)^2} =$$

$$\frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$A(s^2-2s+1) + Bs(s-1) + Cs = 6+s$

$As^2 - 2As + A + Bs^2 - Bs + Cs$

$A+B=0$

$-2A-B+C=1$

$A=6 \quad B=-6$

$-12+6+C=1 \quad C=7$

$y = 6 - 6e^t + 7te^t$