MA 341 Test 3 Version 2

1. (30 points) Use $\vec{\boldsymbol{x}}' = \begin{bmatrix} 0 & -2 & -6 \\ 0 & 2 & 0 \\ 2 & 2 & 8 \end{bmatrix} \vec{\boldsymbol{x}} + \begin{bmatrix} 12e^{8t} \\ 0 \\ 0 \end{bmatrix}$ to answer the following:
   a) Find its complementary solution if its characteristic equation is $(r-2)^2(r-6) = 0$

b) Find a particular solution to $\vec{\boldsymbol{x}}' = \begin{bmatrix} 0 & -2 & -6 \\ 0 & 2 & 0 \\ 2 & 2 & 8 \end{bmatrix} \vec{\boldsymbol{x}} + \begin{bmatrix} 12e^{8t} \\ 0 \\ 0 \end{bmatrix}$ using the **method of undetermined coefficients**. You don't need part a) to do part b).

MA 341 Test 3 Version 2

2. (26 points) Find the particular solution to $\vec{\boldsymbol{x}}' = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \vec{\boldsymbol{x}} + \begin{bmatrix} -6e^{3t} \\ 3e^{3t} \end{bmatrix}$ using

   the **method of variation of parameters** if $\vec{\boldsymbol{x}}_c = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

   Hint: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3. (16 points) Tank A initially holds 97 L of pure water; tank B initially holds 19 L of a brine solution containing 0.2 kg of dissolved salt. The tanks are connected by pipes. The liquid in tank A flows into tank B at a rate of 7 L/min and from tank B into tank A at a rate of 2 L/min. A solution containing 0.3 kg/L of salt is poured into tank A at a rate of 5 L/min and pure water enters tank B at a rate of 1 L/min. Both tanks are well -mixed. The contents of tank B flow out of a drain at the bottom of tank B at a rate of 6 L/min.

   If $x_1(t)$ is the amount of salt in tank A and $x_2(t)$ is the amount of salt in tank B, find the coefficient matrix **A**, $\vec{\mathbf{f}}$, and $\vec{\boldsymbol{x}}(0)$ so that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \mathbf{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \vec{\mathbf{f}}$

   Do not solve this system!

MA 341 Test 3 Version 2

4. (16 points) Find the general solution to the system $\vec{\boldsymbol{x}}' = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix} \vec{\boldsymbol{x}}$

   Hint: If r=$\alpha + \beta i$ with **u**=**a** + **b**i then two linearly independent solutions are $e^{\alpha t}(cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b})$ and $e^{\alpha t}(cos(\beta t)\mathbf{b} + \sin(\beta t)\mathbf{a})$

MA 341 Test 3 Version 2

5. (12 points) Find the inverse of A = $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 3 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix}$ using row operations

MA 341 Test 3 Solutions

1. (30 points)

a) $(A-2I)\vec{u}=\vec{0}$

$$\begin{bmatrix} -2 & -2 & -6 \\ 0 & 0 & 0 \\ 2 & 2 & 6 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}=\vec{0}$$

$$2u_a+2u_b+6u_c=0$$

$$u_a=-u_b-3u_c$$

$$\vec{u_1}=\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \quad \vec{u_2}=\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$(A-6I)\vec{u_3}=\vec{0}$

$$\begin{bmatrix} -6 & -2 & -6 \\ 0 & -4 & 0 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}=\vec{0}$$

$u_b=0 \quad 2u_a+2u_c=0 \quad u_a=-u_c$

$$\vec{u_3}=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x}=c_1e^{2t}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}+c_2e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+c_3e^{6t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

b) $\vec{x_p} = \vec{a} e^{8t}$

$\vec{x_p}' = 8\vec{a} e^{8t}$

$$8\vec{a} e^{8t} = \begin{bmatrix} 0 & -2 & -6 \\ 0 & 2 & 0 \\ 2 & 2 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^{8t} + \begin{bmatrix} 12e^{8t} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8a_1 \\ 8a_2 \\ 8a_3 \end{bmatrix} = \begin{bmatrix} -2a_2 - 6a_3 \\ 2a_2 \\ 2a_1 + 2a_2 + 8a_3 \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8a_1 + 2a_2 + 6a_3 \\ 6a_2 \\ -2a_1 - 2a_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

$a_2 = 0$

$a_1 = 0$

$a_3 = 2$

$$\vec{x_p} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} e^{8t} = \begin{bmatrix} 0 \\ 0 \\ 2e^{8t} \end{bmatrix}$$

2. (26 points)

$$\Phi = \begin{bmatrix} -1 & e^{3t} \\ 2 & e^{3t} \end{bmatrix}$$

$$\Phi^{-1} = \frac{1}{-e^{3t}-2e^{3t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3}e^{-3t} & \frac{1}{3}e^{-3t} \end{bmatrix}$$

$$\vec{x_p} = \Phi \int \Phi^{-1} \vec{f}\, dt$$

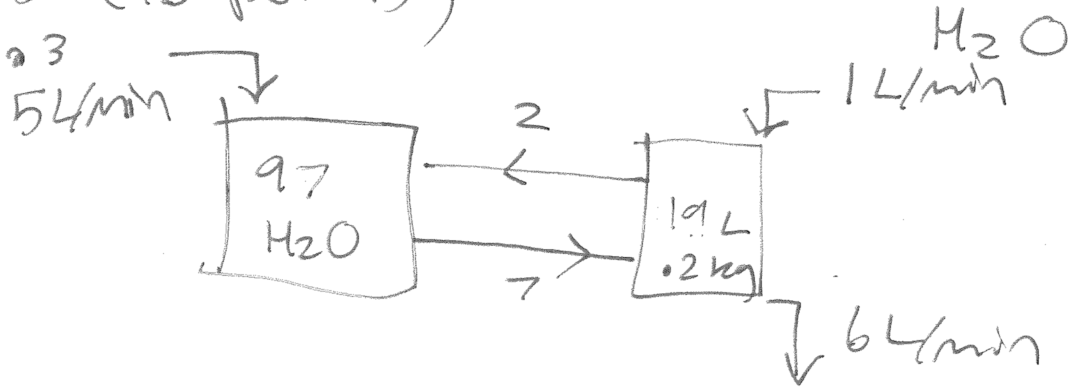
$$= \Phi \int \begin{bmatrix} -1/3 & 1/3 \\ 2/3 e^{-3t} & 1/3 e^{-3t} \end{bmatrix} \begin{bmatrix} -6e^{3t} \\ 3e^{3t} \end{bmatrix} dt$$

$$= \Phi \int \begin{bmatrix} 2e^{3t} + e^{3t} \\ -4 + 1 \end{bmatrix} dt$$

$$= \Phi \int \begin{bmatrix} 3e^{3t} \\ -3 \end{bmatrix} dt$$

$$= \begin{bmatrix} -1 & e^{3t} \\ 2 & e^{3t} \end{bmatrix} \begin{bmatrix} e^{3t} \\ -3t \end{bmatrix} = \begin{bmatrix} -e^{3t} - 3te^{3t} \\ 2e^{3t} - 3te^{3t} \end{bmatrix}$$

3. (16 points)



$$\frac{dx_1}{dT} = F_i C_i - F_o C_o$$

$$= 5(.3) + 2\left(\frac{x_2}{19}\right) - 7\left(\frac{x_1}{97}\right)$$

$$\frac{dx_2}{dT} = 1(0) + 7\left(\frac{x_1}{97}\right) - 8\left(\frac{x_2}{19}\right)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -7/97, & \frac{2}{19} \\ \frac{7}{97} & -\frac{8}{19} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5(.3) \\ 0 \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 0 \\ .2 \end{bmatrix}$$

4. (16 points)

$$\begin{vmatrix} -r & -1 \\ 5 & 2-r \end{vmatrix} = (-r)(2-r) + 5$$

$$r^2 - 2r + 5 = 0$$

$$r = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$$(A - (1+2i)I)\vec{u} = \vec{0}$$

$$\begin{bmatrix} -1-2i & -1 \\ 5 & 1-2i \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix} = \vec{0}$$

$$5u_a + (1-2i)u_b = 0$$

$$u_a = -\frac{1}{5}(1-2i)u_b$$

$$\vec{u} = \begin{bmatrix} 1-2i \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} i$$

$$\vec{x} = c_1 e^t\left(\cos 2t \begin{bmatrix} 1 \\ -5 \end{bmatrix} - \sin 2t \begin{bmatrix} -2 \\ 0 \end{bmatrix}\right) + c_2 e^t\left(\cos 2t \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ -5 \end{bmatrix}\right)$$

5. (12 points)

$$\left[\begin{array}{ccc|ccc} 1/2 & 0 & 1/2 & 1 & 0 & 0 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 0 & 0 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$R_2-3R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -6 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$R_3-2R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -6 & 1 & 0 \\ 0 & 0 & 1 & 12 & -2 & 1 \end{array}\right]$$

$$R_1-R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & 2 & -1 \\ 0 & 1 & 0 & -6 & 1 & 0 \\ 0 & 0 & 1 & 12 & -2 & 1 \end{array}\right]$$

A^{-1}