MA 241 Test 4 Version 1: Be sure to show all of your work and specify every test you use as well as the requirements for each test as we have done in class.

1. (30 points) Determine if the following series converge or diverge. **Find the sum of convergent series.** Justify your answers thoroughly as we have done in class.

   a) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

   b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}} - \frac{1}{\sqrt{n}}$ Include the first three partial sums with your answer

   c) $\sum_{n=0}^{\infty} \frac{8}{(-3)^n}$

2. (6 points) Use $a_n = \frac{1}{n}$ to answer the following:

   a) Does the sequence $a_n = \frac{1}{n}$ converge or diverge? If it converges, find its limit.

   b) Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Briefly justify your answer.

3. (13 points) Determine if the series $\sum_{n=2}^{\infty} \frac{1}{n[\ln(n)]^2}$ converges or diverges by the Integral Test. Briefly mention the two conditions we need to have before we can apply the Integral Test.

4. (24 points) Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n\sqrt{n+3}}$

5. (14 points) a) Find a power series representation for f(x)=$\frac{x}{16-x^4}$ and determine its radius of convergence. Fully simplify your series as we have done in class.

   b) Use your answer from part a) to find $\int \frac{x}{16-x^4}\,dx$

6. (13 points) Determine if the following series are absolutely convergent, conditionally convergent, or divergent

   a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$

   b) $\sum_{n=1}^{\infty} \frac{n^2}{(-5)^n}$

C2 T4 V1 Solutions

1. (30 points)

a) $\lim_{n\to\infty} \cos\frac{1}{n} = \cos 0 = 1 \neq 0$ diverges

Divergence test

b) $S_1 = \frac{1}{\sqrt{3}} - 1$

$S_2 = \frac{1}{\sqrt{3}} - 1 + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{2}}$

$S_3 = \cancel{\frac{1}{\sqrt{3}}} - 1 + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} - \cancel{\frac{1}{\sqrt{3}}}$

$S_4 = -1 + \cancel{\frac{1}{\sqrt{4}}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \cancel{\frac{1}{\sqrt{4}}}$

$S_n = -1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}}$

$\lim_{n\to\infty} S_n = \boxed{-1 - \frac{1}{\sqrt{2}}}$ converges to

Telescoping

c) $8 - \frac{8}{3} + \frac{8}{3^2} - \cdots$ Geometric series

$a + ar + ar^2$

$a = 8 \quad r = -1/3 \quad |-1/3| < 1$

converges to $\frac{a}{1-r} = \frac{8}{1-(-\frac{1}{3})} = \frac{8}{\frac{4}{3}}$

$= \boxed{6}$

2. (6 points)

a) Yes $\lim_{n\to\infty} \frac{1}{n} = 0$

b) Harmonic series diverges

3. (13 points)

$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ positive, decreasing ✓

$u = \ln x \quad du = \frac{1}{x}\,dx$

$\int_{\ln 2}^{\infty} \frac{1}{u^2}\,du = \lim_{t\to\infty} \int_{\ln 2}^{t} \frac{1}{u^2}\,du$

$= \lim_{t\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{t} = \lim_{t\to\infty} -\frac{1}{t} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$

conv.

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by Integral test

4. (24 points) Ratio test

$$\lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{2^{n+1}\sqrt{n+4}} \cdot \frac{2^n\sqrt{n+3}}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-1)\sqrt{n+3}}{2\sqrt{n+4}} \right| = \left| \frac{x-1}{2} \right| < 1$$

$$|x-1| < 2$$

$R = 2$ $\quad a = 1$

$x = a - R \qquad x = a + R$

$x = 1 - 2 = -1 \qquad x = 1 + 2 = 3$

$x = -1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

$c_n = \frac{1}{\sqrt{n+3}} \geq c_{n+1} = \frac{1}{\sqrt{n+4}}$ dec ✓

$\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n+3}} = 0$ ✓ conv AST

$x = 3$: $\displaystyle\sum_{n=1}^{\infty} \frac{\cancel{2^n}}{\cancel{2^n}\sqrt{n+3}}$

LCT $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+3}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+3}} = 1 > 0$ finite

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p-series $p = \frac{1}{2} \not> 1$ diverges

Interval of C: $[-1, 3)$

5. (14 points)

a) $$\frac{x}{16}\left(\frac{1}{1-\frac{x^4}{16}}\right) = \frac{x}{16}\sum_{n=0}^{\infty}\left(\frac{x^4}{16}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{x^{4n+1}}{16^{n+1}}$$

$$\left|\frac{x^4}{16}\right| < 1$$

$$|x^4| < 16$$

$$|x| < 16^{1/4} = 2$$

$$\boxed{R = 2}$$

b) $$\int \sum_{n=0}^{\infty}\frac{x^{4n+1}}{16^{n+1}}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{x^{4n+2}}{16^{n+1}(4n+2)} + C$$

6. (13 points)

a) $\sum_{n=1}^{\infty} \frac{1}{n^3+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$ p-series $p=3>1$ converges

converges comparison test

$\boxed{\text{absolutely convergent}}$

b) Ratio test

$\lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right|$

$\lim_{n\to\infty} \left| \frac{(n+1)^2}{5^{n+1}} \cdot \frac{5^n}{n^2} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{5n^2} \right| = \frac{1}{5} < 1$

$\boxed{\text{absolutely convergent}}$