

1. (15 points) Use the method of undetermined coefficients to solve the Initial Value Problem (IVP): $y'' + 4y' + 3y = 9t$; $y(0)=0$, $y'(0)=1$

$$r^2 + 4r + 3 = 0$$

$$(r+3)(r+1) = 0$$

$$y_c = C_1 e^{-3t} + C_2 e^{-t}$$

$$y_p = At + B$$

$$y_p' = A$$

$$y_p'' = 0$$

$$0 + 4A + 3(At + B) = 9t$$

$$3A = 9 \quad A = 3$$

$$4A + 3B = 0$$

$$12 + 3B = 0 \quad B = -4$$

$$y_p = 3t - 4$$

$$y = C_1 e^{-3t} + C_2 e^{-t} + 3t - 4$$

$$y(0) = 0 = C_1 + C_2 - 4 \quad C_1 + C_2 = 4$$

$$y' = -3C_1 e^{-3t} - C_2 e^{-t} + 3$$

$$y'(0) = 1 = -3C_1 - C_2 + 3 \rightarrow \frac{-3C_1 - C_2}{-2C_1} = \frac{-2}{2}$$

$$\boxed{y = -2e^{-3t} + 6e^{-t} + 3t - 4}$$

$$-2C_1 = 2 \\ C_1 = -2 \quad C_2 = 6$$

2. (14 points) Use the method of variation of parameters to find a particular solution to:

$$y'' - 2y' + y = \frac{e^t}{\sqrt{t}}$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$y_c = c_1 e^t + c_2 t e^t$$

$$y_p = v_1 e^t + v_2 t e^t$$

$$- (v_1' e^t + v_2' t e^t = 0)$$

$$v_1' e^t + v_2' (e^t + t e^t) = \frac{e^t}{\sqrt{t}}$$

$$v_2' e^t = \frac{e^t}{\sqrt{t}}$$

$$v_2' = \frac{1}{\sqrt{t}}$$

$$v_2 = \int \frac{1}{\sqrt{t}} dt$$

$$= 2\sqrt{t}$$

$$v_1' = -v_2' t = -\frac{1}{\sqrt{t}} t = -t^{1/2}$$

$$v_1 = \int -t^{1/2} dt = -\frac{2}{3} t^{3/2}$$

$$y_p = -\frac{2}{3} t^{3/2} e^t + 2\sqrt{t} t e^t$$

3. (13 points) A 16 pound weight attached to the end of a hanging spring stretches the spring 8 ft upon coming to rest at equilibrium. Its damping constant is 3 lb-s/ft. At $t=0$, an external force of 18 N is applied to the system. Hint: If it is needed, use gravity = 32 ft/s^2 to answer the following:
- a) Let $y(t)$ be the position of the weight at the time t . If the weight starts in motion from the equilibrium position with a downward velocity of 2 ft/s, formulate the IVP that describes this system (**Do not solve**)

$$my'' + by' + ky = F \sin t$$

$$\frac{1}{2}y'' + 3y' + 2y = 18$$

$$y(0) = 0 \quad y'(0) = 2$$

$$\text{Weight} = mg$$

$$16 = m(32)$$

$$m = 1/2 \text{ slug}$$

$$16 = k(8) \quad k = 2$$

- b) Find its steady state solution

$$y_p = A \quad y_p' = 0 \quad 2A = 18$$

$$A = 9$$

$$y_p = 9$$

4. (12 points) Use the definition of the Laplace transform to find the Laplace transform of $f(t)=u(t-3)$ and state its domain.

$$\begin{aligned}\mathcal{L}\{u(t-3)\} &= \int_0^{\infty} e^{-st} u(t-3) dt \\ &= \int_0^3 e^{-st} \cdot 0 dt + \int_3^{\infty} e^{-st} \cdot 1 dt \\ &= \lim_{n \rightarrow \infty} \int_3^n e^{-st} dt = \lim_{n \rightarrow \infty} \left. \frac{-1}{s} e^{-st} \right|_3^n \\ &= \lim_{n \rightarrow \infty} \frac{-1}{s} (e^{-sn} - e^{-3s}) = \boxed{\frac{-1}{s} (0 - e^{-3s})}\end{aligned}$$

$$\boxed{s > 0}$$

$L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$	$L\{\cos(bt)\} = \frac{s}{s^2 + b^2}$	$L\{e^{at}\} = \frac{1}{s - a}$
$L\{y'\} = sL\{y\} - y(0)$	$L\{\sin(bt)\} = \frac{b}{s^2 + b^2}$	$L\{t^n\} = \frac{n!}{s^{n+1}}$
$L\{t^n e^{at}\} = \frac{n!}{(s - a)^{n+1}}$	$L\{e^{at} \cos(bt)\} = \frac{s - a}{(s - a)^2 + b^2}$	$L\{e^{at} \sin(bt)\} = \frac{b}{(s - a)^2 + b^2}$
$L\{g(t)u(t - a)\} = e^{-as}L\{g(t + a)\}$	$L^{-1}\{e^{-as}F(s)\} = f(t - a)u(t - a)$	$L\{t^n f(t)\} = (-1)^n \frac{d^n(F(s))}{ds^n}$
	$L\{1\} = \frac{1}{s}$	$L\{u(t - a)\} = \frac{e^{-as}}{s}$

5. (14 points) Express the given function using unit step functions and find its Laplace

$$\text{transform } f(t) = \begin{cases} e^{4t}, & t < 3 \\ 0, & 3 \leq t < 6 \\ 12, & 6 \leq t \end{cases}$$

$$f(t) = e^{4t} - e^{4t}u(t-3) + 12u(t-6)$$

$$\mathcal{L}\{f\} = \frac{1}{s-4} - e^{-3s} \mathcal{L}\{e^{4(t+3)}\} + \frac{12e^{-6s}}{s}$$

$$= \frac{1}{s-4} - e^{-3s} \mathcal{L}\{e^{4t}e^{12}\} + \frac{12e^{-6s}}{s}$$

$$= \frac{1}{s-4} - e^{-3s} \frac{e^{12}}{s-4} + \frac{12e^{-6s}}{s}$$

Use the table below to answer the following:

$L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$	$L\{\cos(bt)\} = \frac{s}{s^2 + b^2}$	$L\{e^{at}\} = \frac{1}{s - a}$
$L\{y'\} = sL\{y\} - y(0)$	$L\{\sin(bt)\} = \frac{b}{s^2 + b^2}$	$L\{t^n\} = \frac{n!}{s^{n+1}}$
$L\{t^n e^{at}\} = \frac{n!}{(s - a)^{n+1}}$	$L\{e^{at} \cos(bt)\} = \frac{s - a}{(s - a)^2 + b^2}$	$L\{e^{at} \sin(bt)\} = \frac{b}{(s - a)^2 + b^2}$
$L\{g(t)u(t - a)\} = e^{-as}L\{g(t + a)\}$	$L^{-1}\{e^{-as}F(s)\} = f(t - a)u(t - a)$	$L\{t^n f(t)\} = (-1)^n \frac{d^n(F(s))}{ds^n}$
	$L\{1\} = \frac{1}{s}$	$L\{u(t - a)\} = \frac{e^{-as}}{s}$

6. (17 points) Find the inverse Laplace of the following: $\frac{10}{s(s^2 - 2s + 5)} e^{-3s}$

$$F(s) = \frac{10}{s(s^2 - 2s + 5)} = \frac{A}{s} + \frac{B(s-1) + 2C}{(s-1)^2 + 4}$$

$$\begin{aligned} & (s-\alpha)^2 + \beta^2 \\ & s^2 - 2\alpha s + \alpha^2 + \beta^2 \\ & \alpha = 1 \quad \beta = 2 \end{aligned}$$

$$A(s^2 - 2s + 5) + (B(s-1) + 2C)s = 10$$

$$\underbrace{As^2 - 2As + 5A} + \underbrace{Bs^2 - Bs + 2Cs} = 10$$

$$A + B = 0$$

$$-2A - B + 2C = 0$$

$$5A = 10 \quad A = 2 \quad B = -2$$

$$-4 + 2 + 2C = 0 \quad C = 1$$

$$F(s) = \frac{2}{s} - \frac{2(s-1)}{(s-1)^2 + 4} + \frac{2 \cdot 1}{(s-1)^2 + 4} \rightarrow f(t) = 2 - 2e^t \cos 2t + e^t \sin 2t$$

$$\text{Ans: } f(t-3)u(t-3) = \left[2 - 2e^{t-3} \cos(2(t-3)) + e^{t-3} \sin(2(t-3)) \right] u(t-3)$$

$L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$	$L\{\cos(bt)\} = \frac{s}{s^2 + b^2}$	$L\{e^{at}\} = \frac{1}{s-a}$
$L\{y'\} = sL\{y\} - y(0)$	$L\{\sin(bt)\} = \frac{b}{s^2 + b^2}$	$L\{t^n\} = \frac{n!}{s^{n+1}}$
$L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$	$L\{e^{at} \cos(bt)\} = \frac{s-a}{(s-a)^2 + b^2}$	$L\{e^{at} \sin(bt)\} = \frac{b}{(s-a)^2 + b^2}$
$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$	$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	$L\{t^n f(t)\} = (-1)^n \frac{d^n(F(s))}{ds^n}$
	$L\{1\} = \frac{1}{s}$	$L\{u(t-a)\} = \frac{e^{-as}}{s}$

7. (15 points) Use the method of Laplace transforms to solve the Initial Value Problem:

$$y'' - 4y' + 4y = 12; y(0)=4, y'(0) = 0$$

$$s^2 L - s y'(0) - y(0) - 4[sL - y(0)] + 4L = \frac{12}{s}$$

$$(s^2 - 4s + 4)L = \frac{12}{s} + 4s - 16$$

$$L = \frac{\frac{12}{s} + 4s - 16}{(s^2 - 4s + 4)} \cdot \frac{s}{s} = \frac{12 + 4s^2 - 16s}{s(s-2)^2}$$

$$= \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$A(s^2 - 4s + 4) + Bs(s-2) + C = 12 + 4s^2 - 16s$$

$$\underbrace{As^2}_{1} - \underbrace{4As}_{-4A} + \underbrace{4A}_{4A} + \underbrace{Bs^2}_{Bs^2} - \underbrace{2Bs}_{-2B} + \underbrace{Cs}_{Cs} =$$

$$A+B=4$$

$$-4A-2B+C=-16$$

$$4A=12 \quad A=3, \quad B=1$$

$$-12-2+C=-16$$

$$L = \frac{3}{s} + \frac{1}{s-2} - \frac{2}{(s-2)^2}$$

$$C=-2$$

$$y = 3 + e^{2t} - 2e^{2t}t$$