Name: Solutions

MA 341-050 Test 1 Version 1

1. (14 points)

a) Solve the Initial Value Problem (IVP): $\frac{dy}{dx} = \frac{y^2}{\sqrt{x}}$, $y(1) = 1$

Write your answer with y as an explicit function of x if possible.

b) Does the Existence and Uniqueness Theorem guarantee that this is a unique solution? Justify your answer as we have done in class.

a) $\int \frac{dy}{y^2} = \int \frac{1}{\sqrt{x}}\, dx$

$-\frac{1}{y} = 2\sqrt{x} + C$

$\frac{1}{y} = -2\sqrt{x} + C_1$

$y = \frac{1}{-2\sqrt{x} + C_1}$

$y(1) = 1 = \frac{1}{-2 + C_1}$ $\quad C_1 = 3$

$$\boxed{y = \frac{1}{3 - 2\sqrt{x}}}$$

b) $f = y^2/\sqrt{x}$ is cont at & around $(1,1)$

$\frac{\partial f}{\partial y} = 2y/\sqrt{x}$ is cont at & around $(1,1)$

Yes

2. (10 points) Find the general solution of the following differential equation. Write your answer with y as an explicit function of x if possible.

$$x\frac{dy}{dx} - y = 2x^2e^{2x}$$

$$\frac{dy}{dx} - \frac{y}{x} = 2xe^{2x}$$

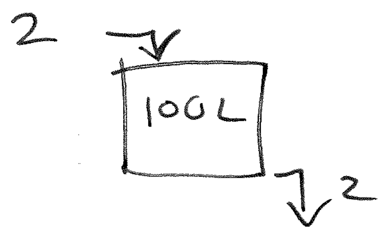
$$\mu = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = x^{-1}$$

$$\frac{d}{dx}\left[x^{-1}y\right] = 2e^{2x}$$

$$x^{-1}y = e^{2x} + C$$

$$\boxed{y = x\left(e^{2x} + C\right)}$$

3. **(6 points)** A tank contains 100 L of liquid. At t=0, brine enters the tank at a rate of 2L/min. The well-stirred liquid leaves the tank at the same rate. If the concentration inside the tank is monitored and found to be $c(t)=\frac{e^{-t/10}}{20}$ answer the following:
   a) Find x(t)
   b) How much salt is initially in the tank?



$$\frac{dx}{dT} = F_i C_i - F_o C_o$$

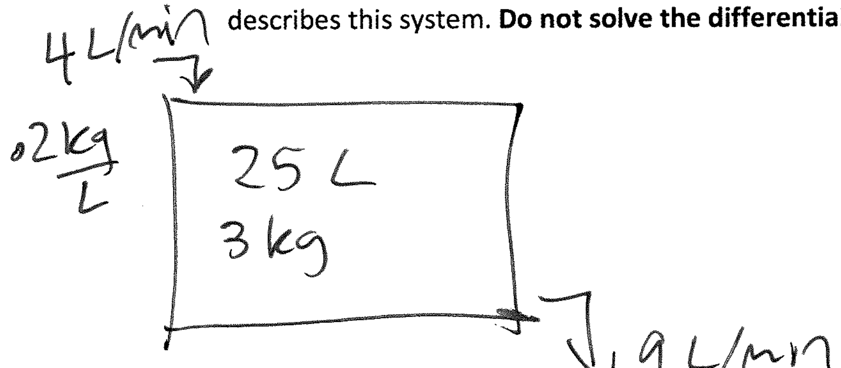
$$\frac{dx}{dt} = 2C_1 - 2\frac{x}{100}$$

$$\frac{x}{100} = \frac{e^{-t/10}}{20}$$

a) $\boxed{x = 5e^{-t/10}}$

b) $x(0) = 5e^{0} = \boxed{5}$

4. (9 points) A large tank initially contains 25 liters of brine in which 3 kg of salt has been dissolved. Brine solution flows into the tank at a rate of 4 L/min. The well-mixed solution leaves the tank at a rate of 9 L/min. If the concentration of salt in the brine entering the tank is 0.2 kg/L and x(t) is the amount of salt in the tank at time t, formulate the IVP that describes this system. **Do not solve the differential equation.**



$$\frac{dx}{dt} = F_i C_i - F_o C_o$$

$$\boxed{\frac{dx}{dT} = 4(.2) - 9\left(\frac{x}{25-5t}\right)} \qquad \boxed{x(0) = 3}$$

5. (14 points)

a) Briefly show that every separable differential equation $\frac{dy}{dx} = f(x)g(y)$ is exact

This is only worth 2 points; don't spend too much time thinking about this if it isn't obvious.

b) Find the value of b that makes $\left(6x^2y^2 + \frac{1}{y}\right)dx + \left(\frac{-x}{y^2} + bx^3y + cos(3y)\right)dy = 0$ an exact differential equation and then solve the differential equation for that value of b

a) $\frac{dy}{g(y)} = f(x)\,dx$

$\underbrace{-f(x)\,dx}_{M} + \underbrace{\frac{1}{g(y)}\,dy}_{N} = 0$

$M_y = N_x = 0$ ✓

b) $M_y = \underline{\underline{12x^2y}} - \frac{1}{y^2} = N_x = \frac{-1}{y^2} + \underline{\underline{3bx^2y}}$

$b = 4$

$F_x = 6x^2y^2 + \frac{1}{y}$

$F = 2x^3y^2 + \frac{x}{y} + g(y)$

$F_y = 4x^3y - \frac{x}{y^2} + g'(y) = \frac{-x}{y^2} + 4x^3y + \cos 3y$

$g'(y) = \cos(3y)$

$g(y) = \frac{1}{3}\sin 3y$

$\boxed{2x^3y^2 + \frac{x}{y} + \frac{1}{3}\sin 3y = C}$

6. (10 points) Solve the Boundary Value Problem (BVP):

$$y'' + 2y' + 26y = 0, \qquad y(0) = 3, \qquad y\left(\frac{\pi}{10}\right) = 0$$

$$r^2 + 2r + 26 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(26)}}{2} = \frac{-2 \pm \sqrt{4(-25)}}{2}$$

$$= \frac{-2 \pm 2 \cdot 5i}{2} = -1 \pm 5i$$

$$y = e^{-t}\left[c_1 \cos 5t + c_2 \sin 5t\right]$$

$$y(0) = \cancel{e^0}\left[c_1 \cancel{\cos 0} + \cancel{c_2 \sin 0}\right] = 3$$

$$c_1 = 3$$

$$y = e^{-t}\left[3\cos 5t + c_2 \sin 5t\right]$$

$$y(\pi/10) = 0 = e^{-\pi/10}\left[\cancel{3\cos \pi/2} + c_2 \sin \pi/2\right]$$

$$c_2 = 0$$

$$\boxed{y = 3e^{-t}\cos 5t}$$

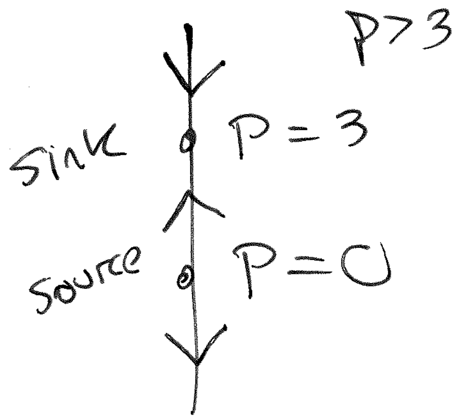
7. (10 points) The logistic equation for the population P (in thousands) of a certain species at time t in years is given by $\frac{dP}{dt} = 6P - 2P^2$

a) Sketch its phase line (include negative values of P) and classify its equilibria as we have done in class

b) If P(12)=3 [the population after 12 years is 3000], what is P(20)?

c) If the initial population is 7000 [P(0)=7], what is the $\lim_{t\to\infty} P(t)$?

$$\frac{dP}{dt} = 2P(3-P)$$



b) $P(20) = 3$

c) $\lim_{t\to\infty} P \to 3$

8. (15 points) Use the method of undetermined coefficients to solve the IVP:
$y'' + 2y' = -24e^{-2t}$; $y(0) = 7,\ y'(0) = 0$

$$\boxed{y = 1 + 6e^{-2t} + 12te^{-2t}}$$

$$r^2 + 2r = 0$$
$$r(r+2) = 0$$
$$r = 0,\ r = -2$$

$$y_c = C_1 + C_2 e^{-2t}$$
$$y_p = Ate^{-2t}$$
$$y_p' = Ae^{-2t} - 2Ate^{-2t}$$
$$y_p'' = -2Ae^{-2t} + 4Ate^{-2t} - 2Ae^{-2t}$$

$$y'' + 2y' = -24e^{-2t}$$

$$-4Ae^{-2t} + \cancel{4Ate^{-2t}} + 2(Ae^{-2t} - \cancel{2Ate^{-2t}}) = -24e^{-2t}$$

$$-2Ae^{-2t} = -24e^{-2t} \qquad A = 12$$

$$y = C_1 + C_2 e^{-2t} + 12te^{-2t}$$
$$y(0) = 7 = C_1 + C_2$$
$$y' = 0 - 2C_2 e^{-2t} + 12e^{-2t} - 24te^{-2t}$$
$$y'(0) = 0 = -2C_2 e^0 + 12e^0 \qquad C_2 = 6 \qquad C_1 = 1$$

9. (12 points) Use the method of variation of parameters to find the particular solution to: $y'' - 2y' + y = \frac{e^t}{t^2+1}$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$y_c = c_1 e^t + c_2 t e^t$$

$$y_p = v_1 e^t + v_2 t e^t$$

$$-\left(v_1' e^t + v_2' t e^t = 0 \right)$$

$$v_1' e^t + v_2' (1e^t + te^t) = \frac{e^t}{t^2+1}$$

$$v_2' e^t = \frac{e^t}{t^2+1} \qquad v_2' = \frac{1}{t^2+1}$$

$$v_2 = \tan^{-1} t$$

$$v_1' = -v_2' t = -\frac{t}{t^2+1}$$

$$v_1 = \int \frac{-t}{t^2+1}\,dt \qquad u = t^2+1 \qquad du = 2t\,dt$$

$$\frac{1}{2}du = t\,dt$$

$$= -\frac{1}{2}\int \frac{1}{u}\,du$$

$$= -\frac{1}{2}\ln u = -\frac{1}{2}\ln(t^2+1)$$

$$\boxed{y_p = -\frac{1}{2}\ln(t^2+1)e^t + (\tan^{-1}t)\,te^t}$$