1. (17 points) Use $\vec{x}' = \begin{bmatrix} 0 & -2 & -6 \\ 0 & 2 & 0 \\ 2 & 2 & 8 \end{bmatrix}\vec{x} + \begin{bmatrix} 12e^{8t} \\ 0 \\ 0 \end{bmatrix}$ to answer the following:

Find its complementary solution, $\vec{x}_c$, if its characteristic equation is $(r-2)^2(r-6) = 0$

$$(A-2I)\vec{u} = \vec{0}$$

$$\begin{bmatrix} -2 & -2 & -6 \\ 0 & 0 & 0 \\ 2 & 2 & 6 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} = \vec{0}$$

$2u_a + 2u_b + 6u_c = 0 \qquad u_a = -u_b - 3u_c$

If $u_b = 0 \quad u_a = -3u_c \qquad \vec{u_1} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

If $u_c = 0 \quad u_a = -u_b \qquad \vec{u_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$(A-6I)\vec{u_3} = \vec{0}$$

$$\begin{bmatrix} -6 & -2 & -6 \\ 0 & -4 & 0 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} = \vec{0} \rightarrow -4u_b = 0$$

$u_b = 0$

$2u_a + 2u_c = 0 \qquad u_a = -u_c$

$\vec{u_3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{\vec{x_c} = c_1 e^{2t}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{6t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

2. (13 points) Find a particular solution, $\vec{\boldsymbol{x}}_p$, to $\vec{\boldsymbol{x}}\,' = \begin{bmatrix} 0 & -2 & -6 \\ 0 & 2 & 0 \\ 2 & 2 & 8 \end{bmatrix} \vec{\boldsymbol{x}} + \begin{bmatrix} 12e^{8t} \\ 0 \\ 0 \end{bmatrix}$ using the **method of undetermined coefficients**. You don't need to solve problem 1 to do problem 2.

$$\vec{x_p} = \vec{a}e^{8t} \qquad \vec{x_p}\,' = 8\vec{a}e^{8t}$$

$$8\vec{a}e^{8t} = \begin{bmatrix} 0 & -2 & -6 \\ 0 & 2 & 0 \\ 2 & 2 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^{8t} + \begin{bmatrix} 12e^{8t} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8a_1 \\ 8a_2 \\ 8a_3 \end{bmatrix} = \begin{bmatrix} -2a_2 - 6a_3 \\ 2a_2 \\ 2a_1 + 2a_2 + 8a_3 \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8a_1 + 2a_2 + 6a_3 \\ 6a_2 \\ -2a_1 - 2a_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} a_2 = 0 \\ a_1 = 0 \\ a_3 = 2 \end{matrix}$$

$$\boxed{\vec{x_p} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} e^{8t}}$$

3. (26 points) Find the particular solution to $\vec{x}' = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} 5e^{-5t} \\ 5e^{-5t} \end{bmatrix}$ using the **method of variation of parameters** if $\vec{x}_c = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Hint: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$X = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & 1e^{-5t} \end{bmatrix}$$

$$X^{-1} = \frac{1}{e^{-5t} + 4e^{-5t}} \begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5}e^{5t} & \frac{1}{5}e^{5t} \end{bmatrix}$$

$$\vec{x_p} = X \int X^{-1} \vec{f}\, dt = X \int \begin{bmatrix} 1/5 & 2/5 \\ -2/5\, e^{5t} & \frac{1}{5}e^{5t} \end{bmatrix} \begin{bmatrix} 5e^{-5t} \\ 5e^{-5t} \end{bmatrix} dt$$

$$= X \int \begin{bmatrix} e^{-5t} + 2e^{-5t} \\ -2 + 1 \end{bmatrix} dt = X \int \begin{bmatrix} 3e^{-5t} \\ -1 \end{bmatrix} dt$$

$$= \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix} \begin{bmatrix} -\frac{3}{5}e^{-5t} \\ -t \end{bmatrix}$$

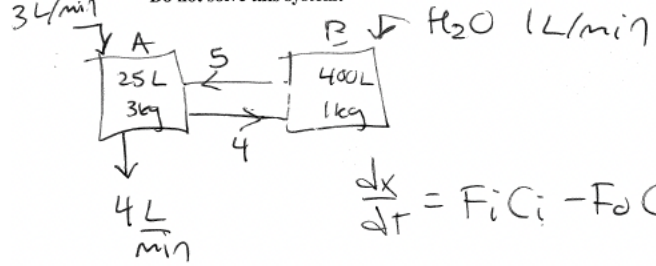
$$= \boxed{\begin{bmatrix} -\frac{3}{5}e^{-5t} - 2te^{-5t} \\ -\frac{6}{5}e^{-5t} - te^{-5t} \end{bmatrix}}$$

4. (16 points) Tank A initially holds 25 L of brine in which 3 kg of salt are dissolved; tank B initially holds 400 L of brine in which 1 kg of salt is dissolved. The tanks are connected by pipes. The liquid in tank A flows into tank B at a rate of 4 L/min and from tank B into tank A at a rate of 5 L/min. A solution containing 0.2 kg/L of salt is poured into tank A at a rate of 3 L/min and pure water enters tank B at a rate of 1 L/min. Both tanks are well -mixed. The contents of tank A flow out of a drain at the bottom of tank a at a rate of 4 L/min.

If $x_1(t)$ is the amount of salt in tank A and $x_2(t)$ is the amount of salt in tank B, find the coefficient matrix **A**, $\vec{\mathbf{f}}$, and $\vec{x}(0)$ so that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \mathbf{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \vec{\mathbf{f}}$

.2kg/L

**Do not solve this system!**




$$\frac{dx}{dT} = F_i C_i - F_o C_o$$

$$\frac{dx_1}{dT} = 3(.2) + 5\left(\frac{x_2}{400}\right) - 8\left(\frac{x_1}{25}\right)$$

$$\frac{dx_2}{dT} = 1(0) + 4\left(\frac{x_1}{25}\right) - 5\left(\frac{x_2}{400}\right)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -8/25 & 5/400 \\ 4/25 & -5/400 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} .6 \\ 0 \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

5: (16 points) Find the general solution to the system $\vec{x}'=\begin{bmatrix}2 & -9\\ 1 & 2\end{bmatrix}\vec{x}$

Hint: If $r=\alpha+\beta i$ with $\mathbf{u}=\mathbf{a}+\mathbf{b}i$ then two linearly independent solutions are $e^{\alpha t}(\cos(\beta t)\mathbf{a}-\sin(\beta t)\mathbf{b})$ and $e^{\alpha t}(\cos(\beta t)\mathbf{b}+\sin(\beta t)\mathbf{a})$

$$|A-rI| = \begin{vmatrix}2-r & -9\\ 1 & 2-r\end{vmatrix} = (2-r)^2+9$$

$$= r^2-4r+13=0$$

$$r=\frac{4\pm\sqrt{16-4(13)}}{2} = \frac{4\pm\sqrt{-36}}{2} = \underset{\alpha}{2}\pm \underset{\beta}{3}i$$

$$(A-(2+3i)I)\vec{u}=\vec{0}$$

$$\begin{bmatrix}-3i & -9\\ 1 & -3i\end{bmatrix}\begin{bmatrix}u_a\\ u_b\end{bmatrix}=\vec{0}$$

$$1u_a-3iu_b=0 \qquad u_a=3iu_b$$

$$\vec{u}=\begin{bmatrix}3i\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}+\begin{bmatrix}3\\ 0\end{bmatrix}i$$

$$\boxed{\vec{x}=c_1e^{2t}\left(\cos 3t\begin{bmatrix}0\\ 1\end{bmatrix}-\sin 3t\begin{bmatrix}3\\ 0\end{bmatrix}\right)+c_2e^{2t}\left(\cos 3t\begin{bmatrix}3\\ 0\end{bmatrix}+\sin 3t\begin{bmatrix}0\\ 1\end{bmatrix}\right)}$$

6. (12 points) Find the inverse of $A=\begin{bmatrix}\frac{1}{3} & 0 & \frac{1}{3}\\ 0 & 1 & 2\\ 2 & 0 & 3\end{bmatrix}$ using row operations

$$[A\,|\,I]=\left[\begin{array}{ccc|ccc}\frac{1}{3} & 0 & \frac{1}{3} & 1 & 0 & 0\\ 0 & 1 & 2 & 0 & 1 & 0\\ 2 & 0 & 3 & 0 & 0 & 1\end{array}\right]$$

$$3R_1\quad\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 3 & 0 & 0\\ 0 & 1 & 2 & 0 & 1 & 0\\ 2 & 0 & 3 & 0 & 0 & 1\end{array}\right]$$

$$R_3-2R_1\quad\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 3 & 0 & 0\\ 0 & 1 & 2 & 0 & 1 & 0\\ 0 & 0 & 1 & -6 & 0 & 1\end{array}\right]$$

$$\begin{array}{l}R_1-R_3\\ R_2-2R_3\end{array}\quad\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 9 & 0 & -1\\ 0 & 1 & 0 & 12 & 1 & -2\\ 0 & 0 & 1 & -6 & 0 & 1\end{array}\right]$$

$A^{-1}$