12.3 Worksheet

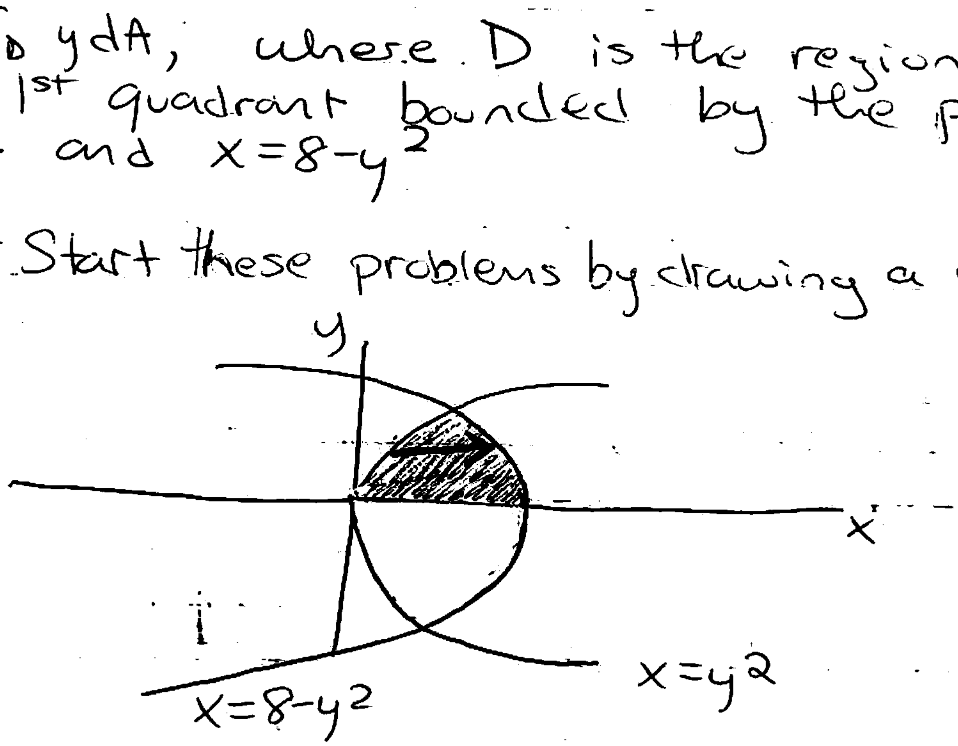
Setting up double integrals, reversing the order of integration, finding volumes

Setting up Double Integrals:

p900

19. $\iint_D y\,dA$, where D is the region in the 1st quadrant bounded by the parabolas $x=y^2$ and $x=8-y^2$

* Start these problems by drawing a picture



* This is what your book refers to as a Type II region (p845). Basically D is between $x=h_1(y)$ and $x=h_2(y)$
For these problems, draw an arrow from the left curve to the right curve.
This gives us our x bounds,
$y^2 \le x \le 8-y^2$

To find y just find the points of intersection of the curves.

$$8-y^2 = y^2$$

$$8 = 2y^2$$

$$4 = y^2 \;<\; y=2,\ \cancel{y=-2}$$ not in the 1st quadrant

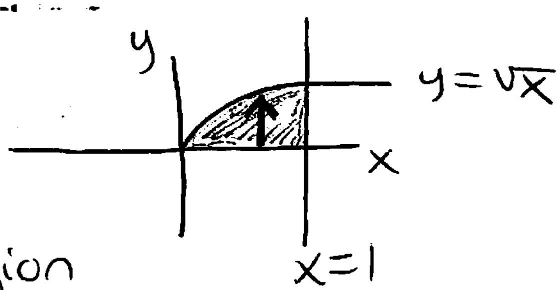
$$\iint_D y\,dA = \int_0^2 \int_{y^2}^{8-y^2} y\,dx\,dy$$

$$= \int_0^2 yx \Big|_{y^2}^{8-y^2} dy$$

$$= \int_0^2 8y - y^3 - y^3 \, dy$$

$$= 4y^2 - \frac{2}{4}y^4 \Big|_0^2 = 16-8 = \boxed{8}$$

$\iint_D \frac{y}{1+x^2}\,dA$ where D is bounded by $y=\sqrt{x}$, $y=0$, $x=1$.



* First draw D.

* This is a Type 1 region and a type 2 region

* First we'll do it as a Type 1 (D is bounded by $y=h_1(x)$ and $y=h_2(x)$)
Draw an arrow from the bottom curve to the top curve, these are your y bounds

$$0 \le y \le \sqrt{x}$$

To find x remember that the outermost limits of integration must be constants. Just find the smallest & largest x can be.

$$\int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx = \int_0^1 \frac{\frac{1}{2}y^2}{1+x^2}\Big|_0^{\sqrt{x}}\,dx$$

$$= \int_0^1 \frac{\frac{1}{2}x}{1+x^2}\,dx$$

$u = 1+x^2 \quad du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

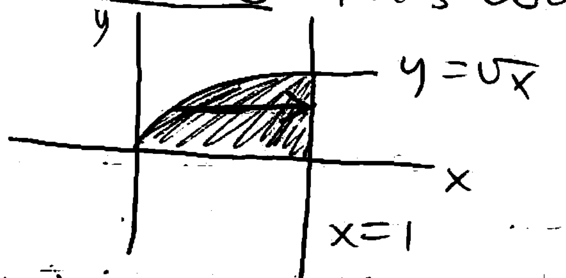
$u(0) = 1+0^2 = 1$

$u(1) = 1+1^2 = 2$

$$= \int_1^2 \frac{\frac{1}{4}}{u}\,du = \frac{1}{4}\ln|u|\,\Big|_1^2 = \frac{1}{4}\ln 2 - \frac{1}{4}\ln 1$$

$$= \boxed{\frac{1}{4}\ln 2}$$

Alternatively, you could have done this problem as a type 2 region. (it is harder this way)



* Draw an arrow going from the left curve to the right curve & solve for x if necessary to get

$y^2 \le x \le 1$

y goes from 0 to $\sqrt{1} = 1$.

$$\int_0^1 \int_{y^2}^1 \frac{y}{1+x^2}\,dx\,dy$$

Reversing the Order of Integration

p850

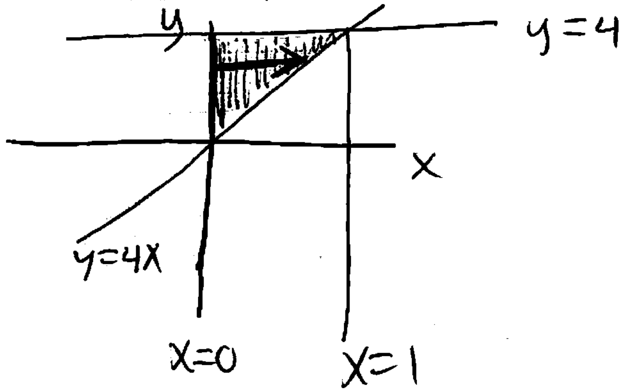
* To change the order of integration just draw D, the integrand doesn't matter!

34. $\int_0^1 \int_{4x}^4 f(x,y)\,dy\,dx$

This tells us that $0 \le x \le 1$ and $4x \le y \le 4$

$\uparrow$ y starts here

$\uparrow$ Stops here



* Since I want to have dxdy, I know x is the inside function. Draw an arrow from the left curve to the right curve & solve for x if necessary

$$0 \le x \le \frac{y}{4}$$

* $0 \le y \le 4$ ← biggest

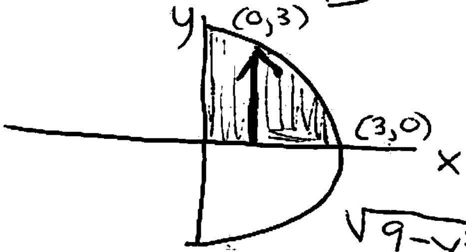
↑ smallest y can be

$$\int_0^1 \int_{4x}^4 f(x,y)\,dy\,dx = \int_0^4 \int_0^{y/4} f(x,y)\,dx\,dy$$

36. $\int_0^3 \int_0^{\sqrt{9-y^2}} f(x,y)\,dx\,dy$

$0 \le y \le 3$ $\quad$ $0 \le x \le \sqrt{9-y^2}$

* Draw D



$\sqrt{9-y^2} = x$

* Draw an arrow from the bottom curve to the top curve and solve for y if necessary

$0 \le y \le \sqrt{9-x^2}$

$0 \le x \le 3$

$$\boxed{\int_0^3 \int_0^{\sqrt{9-x^2}} f(x,y)\,dy\,dx}$$

Volumes in Double Integrals

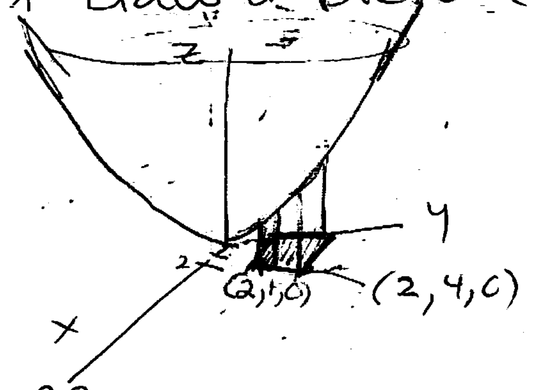
The volume of the solid that lies below the surface given by $z = f(x,y)$ and above the region D in the xy-plane is

$$V = \iint_D f(x,y)\,dA$$

p900 Find the volume of the given solid

29. Under the paraboloid $z = x^2 + 4y^2$ and above the rectangle $R = [0,2] \times [1,4]$

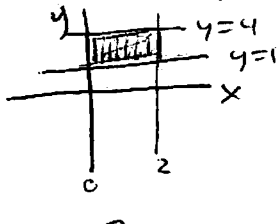
* Draw a picture



$R = [0,2] \times [1,4]$ tells us

$0 \le x \le 2$

$1 \le y \le 4$



$$\int_0^2 \int_1^4 x^2 + 4y^2 \, dy\, dx \quad \text{or} \quad \int_1^4 \int_0^2 x^2 + 4y^2 \, dx\, dy$$

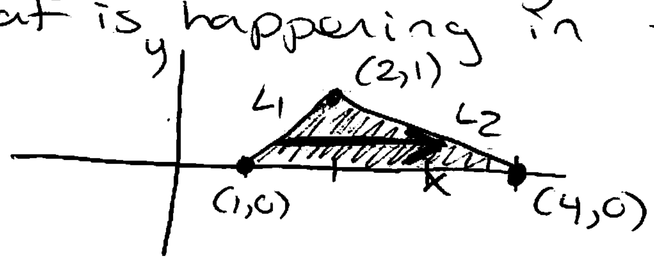
To evaluate this using a triple integral:

$$\int_0^2 \int_1^4 \int_0^{x^2+4y^2} 1 \, dz\, dy\, dx \quad \text{or} \quad \int_1^4 \int_0^2 \int_0^{x^2+4y^2} 1 \, dz\, dx\, dy$$

You can finish integrating these on your own.

30. Find the volume under the surface $z = x^2y$ and above the triangle in the xy-plane with vertices $(1,0)$, $(2,1)$ and $(4,0)$

* We already know $f(x,y) = x^2y$ is our integrand so let's just figure out what is happening in the xy-plane



* This is another Type 2 region; we draw an arrow from the left curve to the right curve.

So there are 2 lines we need to find equations for and then solve for x

$$m_1 = \frac{\Delta y}{\Delta x} = \frac{1-0}{2-1} = 1$$

$y - y_1 = m(x - x_1)$ $\quad$ $y - 0 = 1(x-1)$

$y = x - 1$ is L_1

$$m_2 = \frac{\Delta y}{\Delta x} = \frac{0-1}{4-2} = -\frac{1}{2}$$

$y - y_1 = m(x - x_1)$ $\quad$ $y - 0 = -\frac{1}{2}(x-4)$

$y = -\frac{1}{2}x + 2$ is L_2

* Solve L_1 and L_2 for x to get

$\underbrace{x = y+1}_{L_1}$ and $\underbrace{x = -2y+4}_{L_2}$

$$y+1 \le x \le 4-2y$$

$$0 \le y \le 1$$

$$\int_0^1 \int_{y+1}^{4-2y} x^2 y \, dx \, dy$$ (This is a bit messy when it comes to integrating)