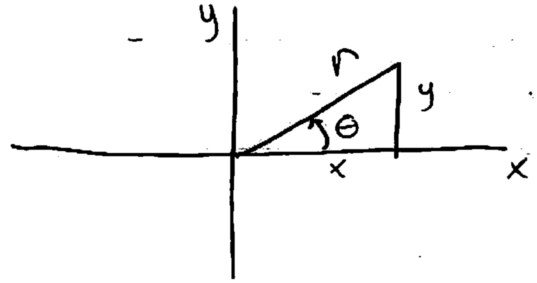
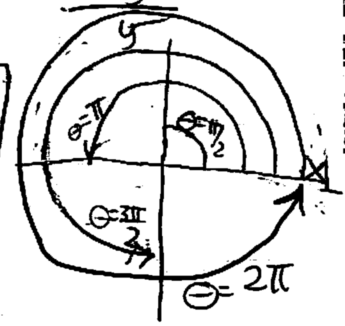
12.4 Polar Coordinates & $\iint$



$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2=r^2$
$dA = r\,dr\,d\theta$



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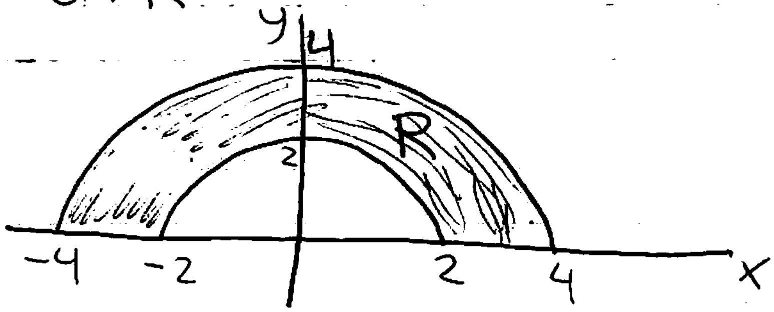
9. Write $\iint_R f(x,y)\,dA$ as an iterated integral where R is the region shown and f is an arbitrary continuous function on R.

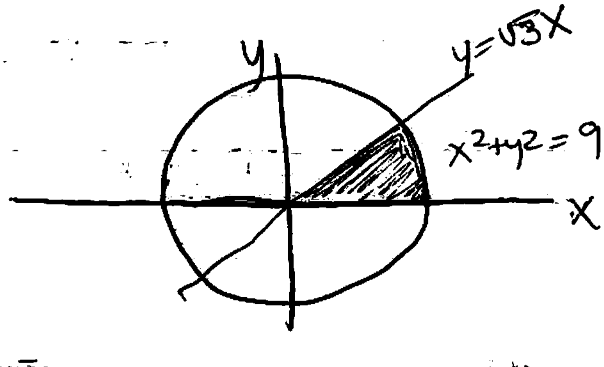


$0 \le \theta \le \pi$
$2 \le r \le 4$

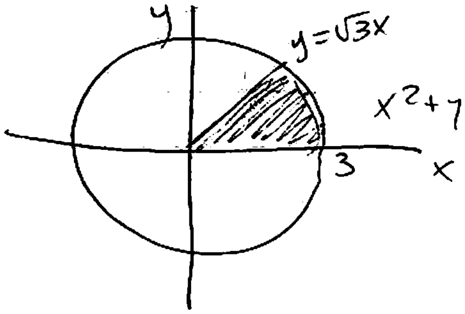
$$\int_0^{\pi}\int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

21. $\iint_D (x^2+y^2)^{3/2}\,dA$ where D is the region in the 1st quadrant bounded by the lines $y=0$, $y=\sqrt{3}x$, and the circle $x^2+y^2=9$

* 1st Draw D



Because our integrand is $(x^2+y^2)^{3/2}$ and our region D is part of a circle it makes a lot of sense to use polar coordinates



$x^2+y^2=9 \quad (r^2=9)$

$(x^2+y^2)^{3/2} = (r^2)^{3/2} = r^3$

* Frequently for polar coordinates θ goes from 0 to 2π, but not here. Determining the bounds of θ will be the hardest part of this problem.

All we need to do is find where $y=\sqrt{3}x$ intersects $x^2+y^2=9$

$$x^2+(\sqrt{3}x)^2=9$$

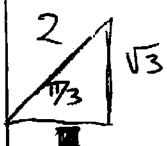
$$x^2+3x^2=9 \qquad 4x^2=9$$

$$x=\frac{3}{2}$$

(we know $x>0$, since we're in the 1st quadrant)

$r\cos\theta=\frac{3}{2}$

or since $r=3$



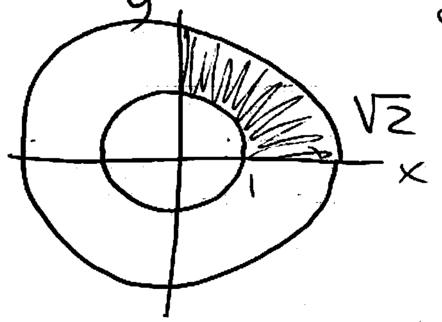
$\cos\theta=\frac{1}{2} \rightarrow \theta=\pi/3$

$$\int_0^{\pi/3}\int_0^3 r^3\, r\,dr\,d\theta=\int_0^{\pi/3}\int_0^3 r^4\,dr\,d\theta$$

$$=\int_0^{\pi/3}\frac{1}{5}r^5\Big|_0^3 d\theta=\int_0^{\pi/3}\frac{3^5}{5}d\theta=\frac{3^5}{5}\theta\Big|_0^{\pi/3}=\boxed{\frac{3^4\pi}{5}}$$

22. $\iint_D x\,dA$ where D is the region in the 1^{st} quadrant that lies between the circles $x^2+y^2=1$ and $x^2+y^2=2$

* Draw D, since it involves circles & double integrals we'll use polar coordinates



$$x^2+y^2=1 \longrightarrow r^2=1 \rightarrow r=1$$

$$x^2+y^2=2 \longrightarrow r^2=2 \rightarrow r=\sqrt{2}$$

$$\int_0^{\pi/2}\int_1^{\sqrt{2}} r\cos\theta \, r\,dr\,d\theta = \int_0^{\pi/2}\int_1^{\sqrt{2}} r^2\cos\theta\,dr\,d\theta$$

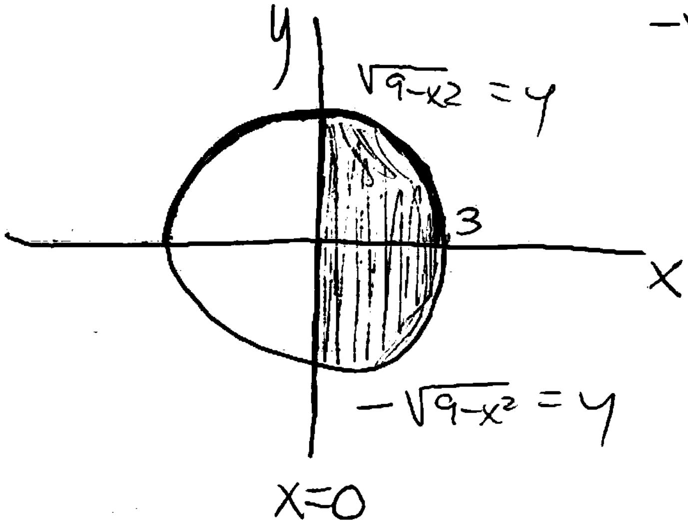
$$= \int_0^{\pi/2} \frac{1}{3}r^3 \Big|_1^{\sqrt{2}} \cos\theta\,d\theta = \int_0^{\pi/2} \frac{1}{3}(2^{3/2}-1)\cos\theta\,d\theta$$

$$= \frac{1}{3}(2^{3/2}-1)\sin\theta \Big|_0^{\pi/2} = \frac{1}{3}(2^{3/2}-1)$$

41. Use polar coordinates to evaluate

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3+xy^2)\,dy\,dx$$

* Just like before we draw D



$-\sqrt{9-x^2} \le y \le \sqrt{9-x^2}$
$0 \le x \le 3$

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\,dy\,dx = \int_{-\pi/2}^{\pi/2} \int_0^3 \left[(r\cos\theta)^3 + r\cos\theta(r\sin\theta)^2\right] r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^3 \left[r^3\cos^3\theta + r\cos\theta\, r^2\sin^2\theta\right] r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^3 r^4\cos^3\theta + r^4\cos\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^3 r^4(\cos\theta)(\cos^2\theta + \sin^2\theta)\,dr\,d\theta$$

$$\int_{-\pi/2}^{\pi/2} \int_0^3 r^4\cos\theta\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{5}r^5\Big|_0^3 \cos\theta\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{243}{5}\cos\theta\,d\theta = \frac{243}{5}\sin\theta\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{243}{5}(1-(-1))$$

$$= \boxed{\frac{486}{5}}$$