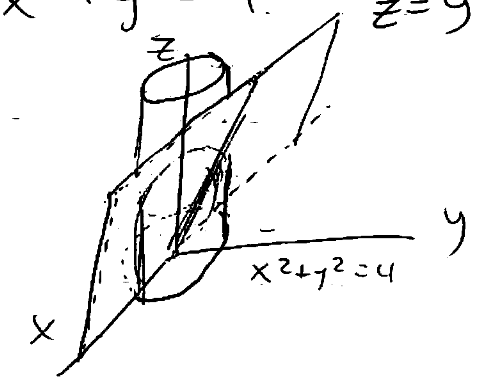
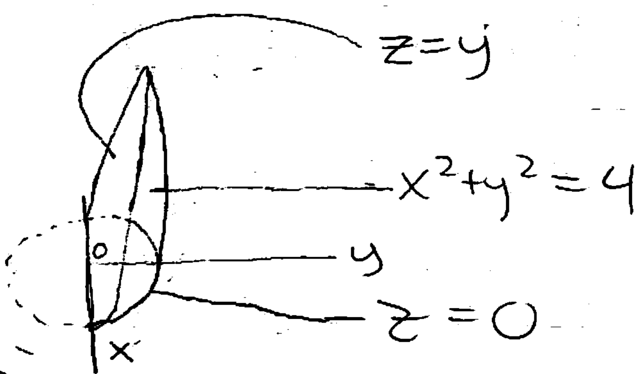
Triple Integrals (12.7 & 12.8)

p900

27. $\iiint_E yz\,dV$, where E lies above the plane $z=0$, below the plane $z=y$ and inside the cylinder $x^2+y^2=4$.



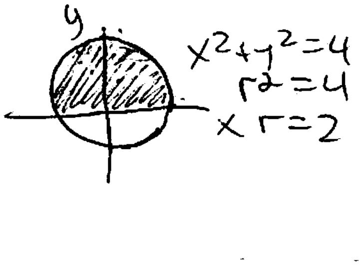
* Draw E



Because this involves a cylinder and z bounds this is a good candidate for cylindrical coordinates

Cylindrical coordinates :
$$\boxed{\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \\ z=z \\ x^2+y^2=r^2 \\ dV=r\,dz\,dr\,d\theta \end{array}}$$

* For cylindrical coordinates we need bounds for θ, r, and z



$z=y$ cuts off 1/2 of our circle in the xy-plane

$0\le\theta\le\pi$

$0\le r\le 2$

$$\int_0^{\pi}\int_0^{2}\int_0^{y} yzr\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} r\sin\theta\, z\, r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} r^2\sin\theta\, z\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2} r^2\sin\theta\,\tfrac{1}{2}z^2\Big|_0^{r\sin\theta} dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2} \tfrac{1}{2} r^4\sin^3\theta\,dr\,d\theta = \int_0^{\pi}\tfrac{1}{10} r^5\Big|_0^{2}\sin^3\theta\,d\theta$$

$$= \int_0^{\pi}\tfrac{16}{5}\sin^3\theta\,d\theta = \int_0^{\pi}\tfrac{16}{5}\sin^2\theta\sin\theta\,d\theta$$

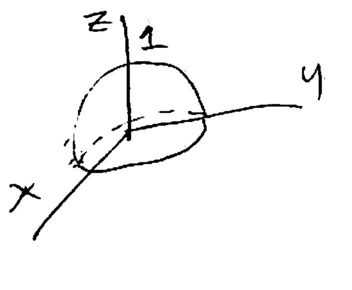
$$= \int_0^{\pi}\tfrac{16}{5}(1-\cos^2\theta)\sin\theta\,d\theta \qquad \begin{array}{ll} u=\cos\theta & u(0)=1 \\ du=-\sin\theta & u(\pi)=-1 \end{array}$$

$$= \int_1^{-1} -\tfrac{16}{5}(1-u^2)\,du = \int_{-1}^{1}\tfrac{16}{5}(1-u^2)\,du = \tfrac{16}{5}\left(u-\tfrac{1}{3}u^3\right)\Big|_{-1}^{1}$$

$$= \tfrac{16}{5}\left(\tfrac{4}{3}\right) = \boxed{\tfrac{64}{15}}$$

28. $\iiint_H z^3\sqrt{x^2+y^2+z^2}\,dV$ where H is the solid hemisphere that lies above the xy-plane and has center the origin and radius 1

* Draw H $\left(z=\sqrt{1-x^2-y^2}\right)$

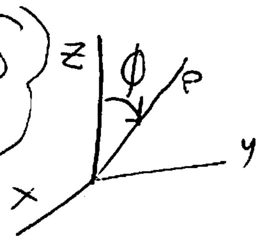


Since we have H is part of a sphere, spherical coordinates would be a good fit.

Spherical Coordinates :

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$
$x^2+y^2+z^2 = \rho^2$
$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Remember: ϕ doesn't go farther than π



$0 \le \theta \le 2\pi$ since we have a whole circle in the xy-plane

$0 \le \phi \le \pi/2$ since we are above the xy-plane

$0 \le \rho \le 1$ since it is a solid hemisphere

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 z^3 \sqrt{x^2+y^2+z^2} \, \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^3 \cos^3\phi \sqrt{\rho^2} \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^6 \cos^3\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{7} \cos^3\phi \sin\phi \, d\phi \, d\theta$$

$u = \cos\phi$
$du = -\sin\phi \, d\phi$
$u(0) = 1 \quad u(\pi/2) = 0$

$$= \int_0^{2\pi}\int_1^0 -\frac{1}{7} u^3 \, du \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{7} u^3 \, du \, d\theta = \int_0^{2\pi} \frac{1}{28} \, d\theta = \boxed{\frac{2\pi}{28}}$$

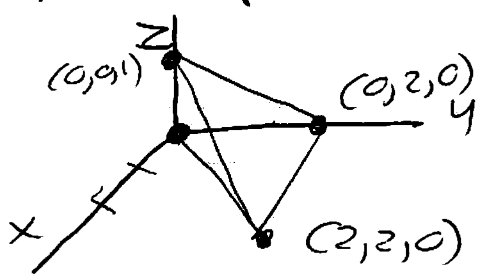
Volumes with Triple Integrals

The volume of a region $E = \iiint_E 1 \, dV$

Notice: If you integrate once you'll get our formula for finding volumes using double integrals.

31. The solid tetrahedron with vertices $(0,0,0)$, $(0,0,1)$, $(0,2,0)$ and $(2,2,0)$

* Graph E



We can see z is between the leaning plane and 0.

1st we need to find the equation for the plane with points $(0,0,1)$, $(0,2,0)$ & $(2,2,0)$

$z = ax + by + c$

$(0,0,1)$

$1 = c \qquad z = ax + by + 1$

$(0,2,0) \quad 0 = 2b + 1 \quad b = -1/2$

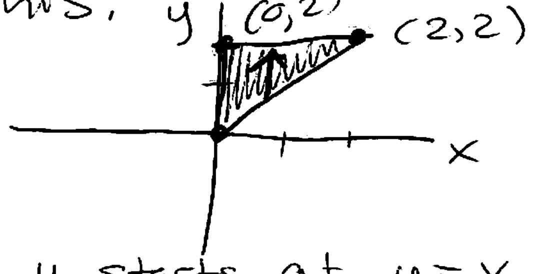
$z = ax - \frac{1}{2}y + 1$

$(2,2,0) \quad 0 = 2a - 1 + 1 \rightarrow a = 0$

$z = 1 - \frac{1}{2}y$

$0 \le z \le 1 - \frac{1}{2}y$

Now we need to describe the xy-plane; it is helpful to draw this.



y starts at $y = x$ (If you don't immediately see this, you could find this by the usual methods) and goes to 2

$$\int_0^2 \int_x^2 \int_0^{1-\frac{1}{2}y} 1 \, dz \, dy \, dx$$

$$= \int_0^2 \int_x^2 z \Big|_0^{1-\frac{1}{2}y} dy \, dx$$

$$= \int_0^2 \int_x^2 1 - \frac{1}{2}y \, dy \, dx = \int_0^2 y - \frac{1}{4}y^2 \Big|_x^2 dx$$

$$= \int_0^2 2 - 1 - \left[x - \frac{1}{4}x^2\right] dx$$

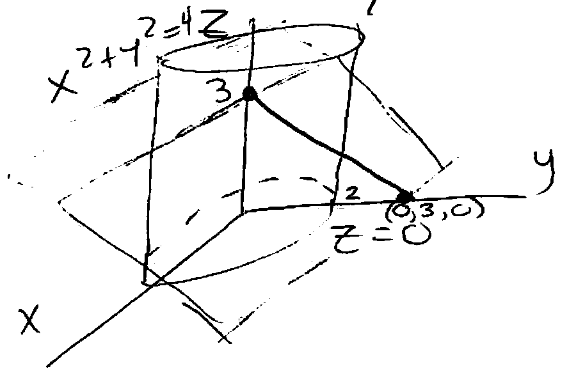
$$= \int_0^2 1 - x + \frac{1}{4}x^2 \, dx = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 \Big|_0^2$$

$$= 2 - 2 + \frac{8}{12}$$

$$= \boxed{\frac{2}{3}}$$

32.

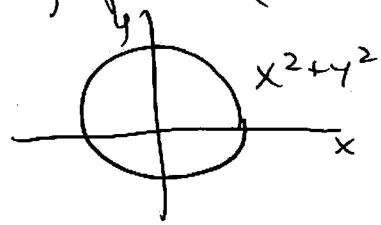
Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $y+z=3$



$z=3-y$ is our top surface
$z=0$ is our bottom surface

Again this is well-suited for cylindrical coordinates $z=3-y=3-r\sin\theta$

We have our whole circle in the xy-plane



$x^2+y^2=4$ $(r^2=4, r=2)$

$$\int_0^{2\pi}\int_0^2\int_0^{3-r\sin\theta} 1\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 z\Big|_0^{3-r\sin\theta} r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (3-r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 3r-r^2\sin\theta\,dr\,d\theta = \int_0^{2\pi}\frac{3}{2}r^2-\frac{1}{3}r^3\sin\theta\Big|_0^2 d\theta$$

$$= \int_0^{2\pi} 6-\frac{8}{3}\sin\theta\,d\theta = 6\theta+\frac{8}{3}\cos\theta\Big|_0^{2\pi}$$

$$= \boxed{12\pi}$$

42. Use spherical coordinates to evaluate

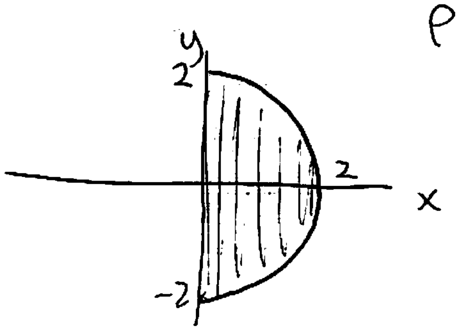
$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

$$-\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2}.$$

Sphere centered at (0,0,0) w/ radius 2

$$x^2+y^2+z^2=4$$
$$\rho^2=4$$

$*$ Draw xy-plane to find θ



$$0 \le x \le \sqrt{4-y^2}$$
$$-2 \le y \le 2$$

$$-\frac{\pi}{2} \le \theta \le \pi/2, \quad 0 \le \phi \le \pi, \quad 0 \le \rho \le 2$$

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{\pi}\int_{0}^{2} \underbrace{(\rho\sin\phi\sin\theta)^2}_{y^2}\sqrt{\rho^2}\,\underbrace{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}_{dz\,dx\,dy}$$

$$=\int_{-\pi/2}^{\pi/2}\int_{0}^{\pi}\int_{0}^{2} \rho^5\sin^3\phi\,\sin^2\theta\,d\rho\,d\phi\,d\theta$$