The Fundamental Theorem for Line Integrals

Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$.
Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

A vector field $\vec{F}$ is <u>conservative</u> if there exists an f (called the <u>potential function</u>) such that $\vec{F} = \nabla f$

* If $\vec{F}(x,y) = P\hat{\imath} + Q\hat{\jmath}$ then $\vec{F}$ is conservative if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

* If $\vec{F}(x,y,z) = P\hat{\imath} + Q\hat{\jmath}\ R\hat{k}$ then $\vec{F}$ is Conservative if $\operatorname{curl}(\vec{F}) = \vec{0}$.

<u>Remember</u>: $\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}$

P931: 7, 9   P946: 12, 14

Determine whether or not $\vec{F}$ is a conservative vector field.
If it is, find a function f such that $\vec{F} = \nabla f$.

7. $\vec{F}(x,y) = (2x\cos y - y\cos x)\hat{\imath} + (-x^2\sin y - \sin x)\hat{\jmath}$

* For $\vec{F}(x,y)$ to be conservative $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$P = 2x\cos y - y\cos x$ $Q = -x^2\sin y - \sin x$

$\frac{\partial P}{\partial y} = -2x\sin y - \cos x$ $\frac{\partial Q}{\partial x} = -2x\sin y - \cos x$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ✓ so $\vec{F}$ is conservative

* To find the potential function f,
set $\vec{F}(x,y) = P\hat{\imath} + Q\hat{\jmath} = \nabla f = f_x\hat{\imath} + f_y\hat{\jmath}$

$f_x = P$ $f_y = Q$

$f_x = 2x\cos y - y\cos x$ $f_y = -x^2\sin y - \sin x$

$f = \int 2x\cos y - y\cos x \, dx$

$f = x^2\cos y - y\sin x + g(y)$

constant relative to x

$f_y = -x^2\sin y - \sin x + g'(y) = -x^2\sin y - \sin x$

$g'(y) = 0$

$g(y) = K$

$\boxed{f = x^2\cos y - y\sin x + K}$

9. $\vec{F}(x,y) = (ye^x + \sin y)\hat{i} + (e^x + x\cos y)\hat{j}$

$P = ye^x + \sin y \qquad Q = e^x + x\cos y$

$\frac{\partial P}{\partial y} = e^x + \cos y \qquad \frac{\partial Q}{\partial x} = e^x + \cos y$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \rightarrow \vec{F}$ is conservative

$f_x = ye^x + \sin y \qquad\qquad f_y = e^x + x\cos y$

integrate both sides with respect to x

$\downarrow$

$f = \int ye^x + \sin y \, dx$

$f = ye^x + x\sin y + g(y)$

differentiate both sides with respect to y

$\downarrow$

$f_y = e^x + x\cos y + g'(y) = e^x + x\cos y$

$g'(y) = 0 \qquad g(y) = K$

$\boxed{f = ye^x + x\sin y + K}$

12. $\vec{F}(x,y,z) = 3z^2 + \cos y\,\hat{j} + 2xz\,\hat{k}$

* $\vec{F}(x,y,z)$ is conservative if $\text{curl}(\vec{F}) = \vec{0}$

$$\text{Curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z^2 & \cos y & 2xz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(2xz) - \frac{\partial}{\partial z}((\cos y)\right)\hat{i} - \left(\frac{\partial}{\partial x}(2xz) - \frac{\partial}{\partial z}(3z^2)\right)\hat{j} +$$

$$\left(\frac{\partial}{\partial x}(\cos y) - \frac{\partial}{\partial y}(3z^2)\right)\hat{k}$$

$$= (0-0)\hat{i} - (2z - 6z)\hat{j} + (0-0)\hat{k}$$

$$\neq \vec{0}$$

$\vec{F}$ is not conservative

14. $\vec{F}(x,y,z) = e^z\,\hat{i} + \hat{j} + xe^z\,\hat{k}$

$$\text{Curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^z & 1 & xe^z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xe^z) - \frac{\partial}{\partial z}(1)\right)\hat{i} - \left(\frac{\partial}{\partial x}(xe^z) - \frac{\partial}{\partial z}(e^z)\right)\hat{j} + \left(\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}e^z\right)\hat{k}$$

$$= (0-0)\hat{i} - (e^z - e^z)\hat{j} + (0-0)\hat{k}$$

$$= \vec{0}$$

so $\vec{F}$ is conservative

$\vec{F}=(x,y,z)=e^{z}\hat{\imath}+\hat{\jmath}+xe^{z}\hat{k}$

$\vec{F}=\nabla f=\langle f_x, f_y, f_z\rangle$

Set $f_x=P$ $f_y=Q$ $f_z=R$

$f_x=e^{z}$ $\qquad$ $f_y=1$ $\qquad$ $f_z=xe^{z}$

integrate with respect to x

$f=\int e^{z}dx$

$f=xe^{z}+g(y,z)$

differentiate with respect to y

$f_y=0+g_y(y,z)=1$

$g_y(y,z)=1$

$g(y,z)=\int 1\,dy$

$=y+h(z)$

* Plug back into f

$f=xe^{z}+y+h(z)$

differentiate with respect to z

$f_z=xe^{z}+0+h'(z)=xe^{z}$

$h'(z)=0$

$h(z)=K$

$$\boxed{f=xe^{z}+y+K}$$

p 932: 12, 16

- Show $\vec{F}$ is conservative
- Find f such that $\vec{F} = \nabla f$
- Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the given curve C

12. $\vec{F}(x,y) = y\hat{i} + (x+2y)\hat{j}$

C is the upper semicircle that starts at (0,1) and ends at (2,1)

$\frac{\partial P}{\partial y} = 1$ $\frac{\partial Q}{\partial x} = 1$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ so $\vec{F}$ is conservative

$f_x = P$ $f_y = Q$

$f_x = y$ $f_y = x + 2y$

$f = \int y\,dx$

$f = xy + g(y)$

$f_y = x + g'(y) = x + 2y$

$g'(y) = 2y$

$g(y) = y^2 + K$

$$\boxed{f = xy + y^2 + K}$$

* Use the Fundamental Theorem to find

$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$= f(2,1) - f(0,1)$

$= 2(1) + 1^2 + K - [0(1) + 1^2 + K]$

$= \boxed{2}$ * Note: The path from a to b doesn't matter if $\vec{F}$ is conservative.

16 $\vec{F}(x,y,z) = (2xz + y^2)\hat{i} + 2xy\hat{j} + (x^2+3z^2)\hat{k}$

$C: x = t^2,\ y = t+1,\ z = 2t-1 \quad 0 \le t \le 1$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz+y^2 & 2xy & x^2+3z^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(x^2+3z^2) - \frac{\partial}{\partial z}(2xy)\right)\hat{i} - \left(\frac{\partial}{\partial x}(x^2+3z^2) - \frac{\partial}{\partial z}(2xz+y^2)\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(2xz+y^2)\right)\hat{k}$$

$$= (0-0)\hat{i} - (2x-2x)\hat{j} + (2y-2y)\hat{k}$$

$= \vec{0}$ so $\vec{F}$ is conservative ✓

$\vec{F} = \nabla f$

$f_x = 2xz + y^2$ $\qquad$ $f_y = 2xy$ $\qquad$ $f_z = x^2+3z^2$

$f = \int 2xz + y^2$

$f = x^2z + xy^2 + g(y,z)$

$f_y = 0 + 2xy + g_y(y,z) = 2xy$

$g_y(y,z) = 0$

$g(y,z) = h(z)$

$f = x^2z + xy^2 + h(z)$

$f_z = x^2 + 0 + h'(z) = x^2 + 3z^2$

$h'(z) = 3z^2$

$h(z) = z^3 + K$

$$\boxed{f = x^2z + xy^2 + z^3 + K}$$

$C: \; x = t^2 \quad y = t+1 \quad z = 2t-1 \quad 0 \le t \le 1$

Since $\vec{F}$ is conservative $\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ by the Fundamental theorem of line Integrals.

$f = x^2 z + xy^2 + z^3 + K$

For evaluating $\int_C \vec{F} \cdot d\vec{r}$ we don't need the $+K$, It will always cancel out like in the last problem

$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

(★ Basically they just want f evaluated at the end point − f evaluated at the start of the curve)

$= f(\vec{r}(1)) - f(\vec{r}(0))$

$\vec{r}(t) = t^2\hat{i} + (t+1)\hat{j} + (2t-1)\hat{k}$

$\vec{r}(1) = \langle 1, 2, 1 \rangle$

$\vec{r}(0) = \langle 0, 1, -1 \rangle$

$= f(1,2,1) - f(0,1,-1)$

$= 1^2(1) + 1 \cdot 2^2 + 1^3 - [0^2(-1) + 0(1)^2 + (-1)^3]$

$= 1 + 4 + 1 - [-1]$

$= \boxed{7}$

p 932 Find the work done by the force field $\vec{F}$ in moving an object from P to Q

21. $\vec{F}(x,y) = 2y^{3/2}\hat{\imath} + 3x\sqrt{y}\,\hat{\jmath}$ $P(1,1), Q(2,4)$

* Notice: If $\vec{F}$ is not conservative then we will need more information to do this problem *

$$\frac{\partial P}{\partial y} = 3y^{1/2} \qquad \frac{\partial Q}{\partial x} = 3\sqrt{y}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \checkmark$$

* **Remember**: $W = \int_C \vec{F} \cdot d\vec{r}$ we can evaluate this using the Fundamental Theorem since $\vec{F}$ is conservative for this problem.

1st we need to find the potential function f

$f_x = 2y^{3/2}$ $\qquad\qquad f_y = 3x\sqrt{y}$

$f = \int 2y^{3/2}\,dx$

$f = 2xy^{3/2} + g(y)$

$f_y = 3x\sqrt{y} + g'(y) = 3x\sqrt{y}$

$g'(y) = 0$

$g(y) = K$

$f = 2xy^{3/2} + K$

$$W = \int_C \vec{F} \cdot d\vec{r} = f(2,4) - f(1,1) = 2(2)4^{3/2} - 2(1)1^{3/2}$$

$$= 32 - 2 = \boxed{30}$$