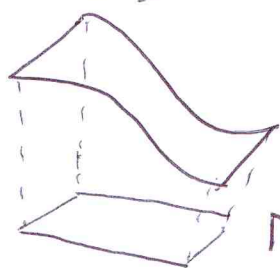


13.6 Surface Integrals

S , surface



D (frequently in the xy -plane)

$$\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| dA$$

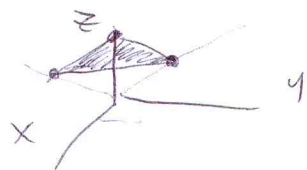
If $x=x, y=y, z=g(x, y)$, this becomes

$$\iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA$$

* If $f = \rho(x, y, z)$ density of the surface, then $\iint_S \rho(x, y, z) dS$ will give us the mass of a thin surface.

p958

7. $\iint_S x^2 y z dS$ S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$

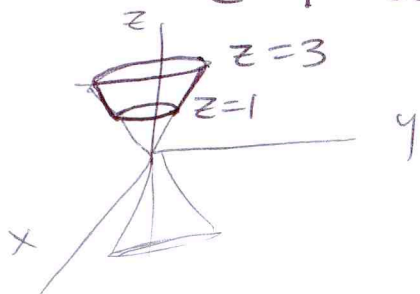


$$\begin{aligned} |\vec{r}_x \times \vec{r}_y| &= \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \\ &= \sqrt{4 + 9 + 1} = \sqrt{14} \end{aligned}$$

(2)

$$\begin{aligned}
& \int_0^3 \int_0^2 x^2 y (1+2x+3y) \sqrt{14} \, dy \, dx \\
&= \int_0^3 \int_0^2 (x^2 y + 2x^3 y + 3x^2 y^2) \sqrt{14} \, dy \, dx \\
&= \int_0^3 \left. \frac{x^2 y^2}{2} + x^3 y^2 + x^2 y^3 \right|_0^2 \sqrt{14} \, dx \\
&= \int_0^3 (2x^2 + 4x^3 + 8x^2) \sqrt{14} \, dx \\
&= \left(\frac{2}{3} x^3 + x^4 + \frac{8}{3} x^3 \right) \Big|_0^3 \sqrt{14} \\
&= (18 + 81 + 72) \sqrt{14} = 171 \sqrt{14}
\end{aligned}$$

11. $\iint_S x^2 z^2 \, dS$ S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z=1$ and $z=3$



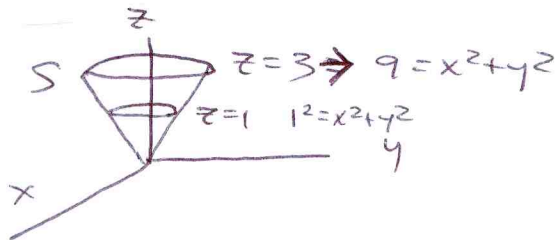
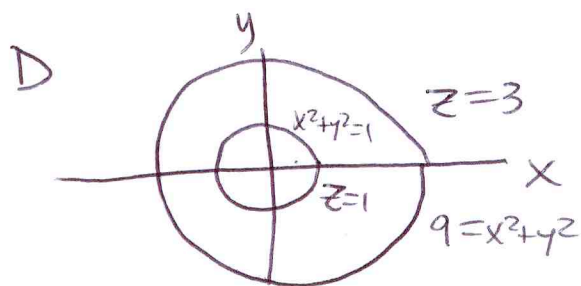
$$z = \sqrt{x^2 + y^2}$$

$$\begin{aligned}
|\vec{r}_x \times \vec{r}_y| &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \\
&= \sqrt{1 + \left(\frac{2x}{2\sqrt{x^2+y^2}}\right)^2 + \left(\frac{2y}{2\sqrt{x^2+y^2}}\right)^2} \\
&= \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} \\
&= \sqrt{1 + \frac{(x^2+y^2)}{(x^2+y^2)}} \\
&= \sqrt{2}
\end{aligned}$$

(3)

$$\iint_S x^2 z^2 dS = \iint_D x^2 (\sqrt{x^2+y^2})^2 \sqrt{2} dA$$

$$= \iint_D x^2 (x^2+y^2) \sqrt{2} dA$$



Polar coordinates work best to describe D
 $x = r \cos \theta$ $y = r \sin \theta$

$$\int_0^{2\pi} \int_1^3 \sqrt{2} r^2 \cos^2 \theta (r^2) r dr d\theta$$

$$= \int_0^{2\pi} \int_1^3 \sqrt{2} r^5 \cos^2 \theta dr d\theta = \int_0^{2\pi} \left. \frac{\sqrt{2} r^6}{6} \right|_1^3 \cos^2 \theta d\theta$$

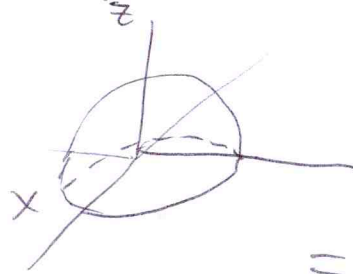
$$= \int_0^{2\pi} \sqrt{2} \left(\frac{3^6 - 1^6}{6} \right) \cos^2 \theta d\theta$$

$$= \int_0^{2\pi} \sqrt{2} \left(\frac{3^6 - 1}{6} \right) \frac{1}{2} (1 + \cos 2\theta) d\theta$$

$$= \frac{\sqrt{2}(3^6 - 1)}{6} \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{2\pi}$$

$$= \boxed{\frac{\sqrt{2}(3^6 - 1) \pi}{6}}$$

15. $\iint_S (x^2z + y^2z) dS$ S is the hemisphere $x^2 + y^2 + z^2 = 4$ $z \geq 0$ (4)



$$z \geq 0 \rightarrow z = \sqrt{4 - x^2 - y^2}$$

$$\left| \vec{r}_x \times \vec{r}_y \right| = \sqrt{\left(\frac{-2x}{2\sqrt{4-x^2-y^2}} \right)^2 + \left(\frac{-2y}{2\sqrt{4-x^2-y^2}} \right)^2 + 1}$$

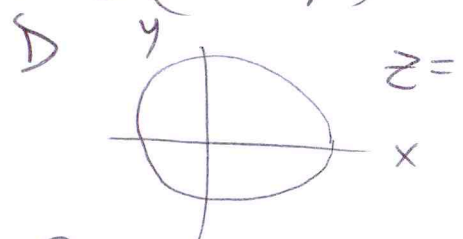
$$= \sqrt{\frac{x^2 + y^2}{4 - x^2 - y^2} + 1}$$

$$= \sqrt{\frac{x^2 + y^2 + 4 - x^2 - y^2}{4 - x^2 - y^2}}$$

$$= \sqrt{\frac{4}{4 - x^2 - y^2}} = \frac{2}{\sqrt{4 - x^2 - y^2}}$$

$$\iint_S x^2z + y^2z dS = \iint_D \left(x^2\sqrt{4-x^2-y^2} + y^2\sqrt{4-x^2-y^2} \right) \frac{2}{\sqrt{4-x^2-y^2}} dA$$

$$= \iint_D (x^2 + y^2) 2 dA \quad * \text{Now describe } D *$$



$z=0$
 $x^2 + y^2 = 4$
 Convert to polar coordinates
 $x = r \cos \theta$ $y = r \sin \theta$

$$= \int_0^{2\pi} \int_0^2 2r^2 r dr d\theta = \int_0^{2\pi} \int_0^2 2r^3 dr d\theta$$

$$= \int_0^{2\pi} \frac{2}{4} r^4 \Big|_0^2 d\theta = \int_0^{2\pi} 2(4) d\theta = \boxed{16\pi}$$

Surface Integrals of Vector Fields

⑤

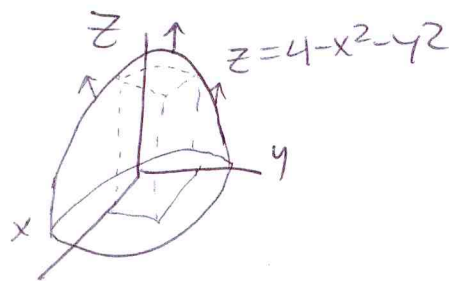
Given a vector field with normal $\vec{n}$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$$

p958

19. $\vec{F}(x, y, z) = xy\hat{i} + yz\hat{j} + zx\hat{k}$,

S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ and has upward orientation



* Just like before we can use a short cut if we have that our surface is

given by $x = x, y = y, z = g(x, y)$

$$\vec{r}_x \times \vec{r}_y = \left(-\frac{\partial z}{\partial x} \hat{i} - \frac{\partial z}{\partial y} \hat{j} + \hat{k} \right) \quad \left(\text{Check this by computing the cross product} \right)$$

$$\begin{aligned} \text{So } \vec{r}_x \times \vec{r}_y &= -(-2x)\hat{i} - (-2y)\hat{j} + \hat{k} \\ &= 2x\hat{i} + 2y\hat{j} + \hat{k} \end{aligned}$$

We have upward orientation so we want positive $\hat{k}$.

(6)

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) dA$$

$$= \iint_D \langle xy, yz, zx \rangle \cdot \langle 2x, 2y, 1 \rangle dA$$

* Note: we need to get rid of those z's. We can do this now or after the dot product *

$$= \iint_D 2x^2y + 2y^2z + zx dA$$

* We know $z = 4 - x^2 - y^2$ so this becomes

$$\begin{aligned} & \iint_D 2x^2y + 2y^2(4 - x^2 - y^2) + (4 - x^2 - y^2)x dA \\ &= \iint_D 2x^2y + 8y^2 - 2y^2x^2 - 2y^4 + 4x - x^3 - xy^2 dA \end{aligned}$$

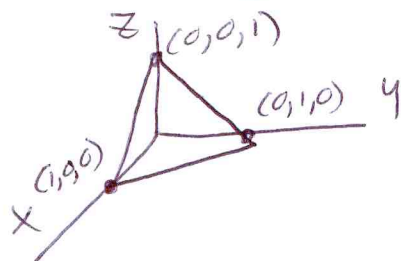
They have already told us D , $0 \leq x \leq 1$, $0 \leq y \leq 1$
so

$$\begin{aligned} & \int_0^1 \int_0^1 2x^2y + 8y^2 - 2y^2x^2 - 2y^4 + 4x - x^3 - xy^2 dy dx \\ &= \int_0^1 x^2y^2 + \frac{8}{3}y^3 - \frac{2}{3}y^3x^2 - \frac{2}{5}y^5 + 4xy - x^3y - \frac{x}{3}y^3 \Big|_0^1 dx \\ &= \int_0^1 x^2 + \frac{8}{3} - \frac{2}{3}x^2 - \frac{2}{5} + 4x - x^3 - \frac{x}{3} dx \\ &= \int_0^1 \frac{1}{3}x^2 + \frac{34}{15} + \frac{11}{3}x - x^3 dx = \left[\frac{1}{9}x^3 + \frac{34}{15}x + \frac{11}{6}x^2 - \frac{1}{4}x^4 \right]_0^1 \\ &= \frac{1}{9} + \frac{34}{15} + \frac{11}{6} - \frac{1}{4} = \boxed{\frac{713}{180}} \end{aligned}$$

(7)

$$21. \vec{F}(x, y, z) = xze^y \hat{j} - xze^y \hat{j} + z \hat{k}$$

S is the part of the plane $x+y+z=1$ in the 1st octant and has downward orientation.



To describe S , just solve $x+y+z=1$ for z , $z=1-x-y$

$$\begin{aligned} \vec{r}_x \times \vec{r}_y &= -\frac{\partial z}{\partial x} \hat{i} - \frac{\partial z}{\partial y} \hat{j} + \hat{k} \\ &= -(-1)\hat{i} - (-1)\hat{j} + \hat{k} = \hat{i} + \hat{j} + \hat{k} \end{aligned}$$

* But we have downward orientation so we need $-\hat{k}$

$$-\vec{r}_x \times \vec{r}_y = -\hat{i} - \hat{j} - \hat{k}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \langle xze^y, -xze^y, z \rangle \cdot \langle -1, -1, -1 \rangle dA$$

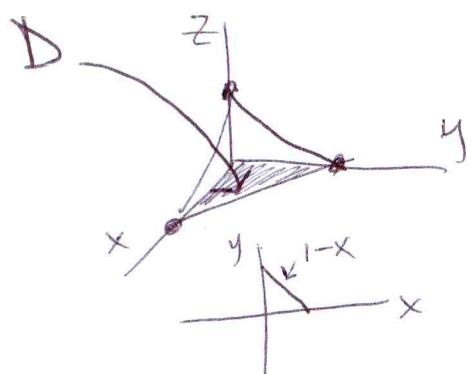
$$= \iint_D -xze^y + xze^y - z dA = \iint_D -z dA$$

* Eliminate z

$$= \iint_D -(1-x-y) dA = \iint_D x+y-1 dA$$

(8)

$\iint_D -1+x+y \, dA$ we need to describe



* Again for D, $z=0$

so if $z=1-x-y$

$0=1-x-y$

$y=1-x$

$$\int_0^1 \int_0^{1-x} -1+x+y \, dy \, dx = \int_0^1 -y + yx + \frac{1}{2}y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 -1+x + x - x^2 + \frac{1}{2}(1-x)^2 dx$$

$$= \int_0^1 -1+x + \cancel{x} - x^2 + \frac{1}{2} \cancel{x} + \frac{x^2}{2} dx$$

$$= \int_0^1 x - \frac{1}{2} - \frac{x^2}{2} dx = \frac{1}{2}x^2 - \frac{1}{2}x - \frac{x^3}{6} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{2} - \frac{1}{6} = \boxed{-\frac{1}{6}}$$

p976

27. $\iint_S \vec{F} \cdot d\vec{S}$ $\vec{F}(x,y,z) = xz\hat{i} - 2y\hat{j} + 3x\hat{k}$
 S is the sphere $x^2+y^2+z^2=4$ with

outward orientation,

We want the whole sphere so
 we can represent S as

$$x = 2\sin\phi\cos\theta, \quad y = 2\sin\phi\sin\theta, \quad z = 2\cos\phi$$

(9)

$$\text{So } \vec{r} = 2\sin\phi\cos\theta\hat{i} + 2\sin\phi\sin\theta\hat{j} + 2\cos\phi\hat{k}$$

$$\vec{r}_\theta = \langle -2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0 \rangle$$

$$\vec{r}_\phi = \langle 2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi \rangle$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= 4\sin^2\phi\cos\theta\hat{i} + 4\sin^2\phi\sin\theta\hat{j} + (4\cos\phi\sin\phi\cos^2\theta + 4\sin^2\theta\cos\phi)\hat{k}$$

$$= \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\cos\phi\sin\phi \rangle$$

$$\vec{F}(\vec{r}(\theta, \phi)) = 4\sin\phi\cos\theta\cos\phi\hat{i} - 4\sin\phi\sin\theta\hat{j} + 6\sin\phi\cos\phi\hat{k}$$

$$\vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta) = 16\sin^3\phi\cos^2\theta\cos\phi - 16\sin^3\phi\sin^2\theta + 24\sin^2\phi\cos\theta\cos\phi$$

$$\int_0^{2\pi} \int_0^\pi \vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta) d\phi d\theta$$

$$= -\frac{64}{3}\pi \text{ after many trig integrals}$$

