

P1

## 9.7 Variation of Parameters

General solution to  $\vec{x}' = A\vec{x} + \vec{f}$  is

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

where

$\vec{x}_c$  is the solution to  $\vec{x}' = A\vec{x}$  and

$$\vec{x}_p = \mathbf{X} \int \mathbf{X}^{-1} \vec{f} dt$$

Examples from Advanced Engineering Mathematics by Zill & Cullen p594

**Ex 1** Find the general solution

$$\vec{x}' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix} \vec{x} + \begin{pmatrix} 12 \\ 12 \end{pmatrix} t$$

\* 1st find  $\vec{x}_c$

$$|A - rI| = 0$$

$$\begin{vmatrix} 1-r & 8 \\ 1 & -1-r \end{vmatrix} = (1-r)(-1-r) - 8 = 0$$

$$r^2 - 9 = 0$$

$$r = \pm 3 \quad r_1 = 3, r_2 = -3$$

$$(A - 3I)\vec{u} = \vec{0}$$

$$\begin{pmatrix} -2 & 8 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u_1 = 4u_2 \quad \vec{u}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}_s$$

P2

$$(A - (-3)I)\vec{u} = \vec{0}$$

$$\begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix} \vec{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u_1 + 2u_2 = 0$$

$$u_1 = -2u_2$$

$$\vec{u}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}_s$$

$$\vec{X}_c = c_1 e^{3t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}$$

$$\mathbb{X} = \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix}$$

Recall:  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$   $A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$  if  $\det A \neq 0$

$$\mathbb{X}^{-1} = \frac{1}{4+2} \begin{pmatrix} e^{-3t} & 2e^{-3t} \\ -e^{3t} & 4e^{3t} \end{pmatrix} = \begin{pmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{pmatrix}$$

$$\vec{X}_p = \mathbb{X} \int \mathbb{X}^{-1} \vec{F} dt$$

$$= \mathbb{X} \int \begin{pmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{pmatrix} \begin{pmatrix} 12t \\ 12t \end{pmatrix} dt$$

$$= \mathbb{X} \int \begin{pmatrix} 2te^{-3t} + 4te^{-3t} \\ -2te^{3t} + 8te^{3t} \end{pmatrix} dt = \mathbb{X} \int \begin{pmatrix} 6te^{-3t} \\ 6te^{3t} \end{pmatrix} dt$$

Integration By parts  $uv - \int v du$

$$= \mathbb{X} \begin{pmatrix} \frac{6t}{-3}e^{-3t} - \int \frac{6}{-3}e^{-3t} dt \\ \frac{6t}{3}e^{3t} - \int \frac{6}{3}e^{3t} dt \end{pmatrix}$$

$$= \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \begin{pmatrix} (-2te^{-3t} - \frac{2}{3}e^{-3t}) \\ (2te^{3t} - \frac{2}{3}e^{3t}) \end{pmatrix} = \begin{pmatrix} -8t - \frac{8}{3} - 4t + \frac{4}{3} \\ -2t - \frac{2}{3} + 2t - \frac{2}{3} \end{pmatrix}$$

$$\vec{X} = c_1 e^{3t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} -12t - \frac{4}{3} \\ -\frac{4}{3} \end{pmatrix}$$

p3

Ex 2

$$\vec{X}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{X} + \begin{pmatrix} \sec t \\ 0 \end{pmatrix}$$

$$|A - rI| = 0$$

$$\begin{vmatrix} -r & -1 \\ 1 & -r \end{vmatrix} = r^2 + 1 = 0 \quad r = \pm i \quad \leftarrow \text{complex}$$

$$(A - iI)\vec{u} = \vec{0}$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u_1 = u_2 i \quad \vec{u} = \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} i$$

$\vec{a}$                        $\vec{b}$

$$\begin{aligned} \vec{X}_c &= C_1 e^{it} \left( \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + C_2 e^{it} \left( \cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= C_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + C_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \end{aligned}$$

$$\vec{X} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix}$$

$$\begin{aligned} \vec{X}^{-1} &= \frac{1}{-\sin^2 t - \cos^2 t} \begin{pmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{pmatrix} = -1 \begin{pmatrix} \sin t & -\cos t \\ -\cos t & -\sin t \end{pmatrix} \\ &= \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \vec{X}_p &= \vec{X} \int \vec{X}^{-1} \vec{f} dt = \vec{X} \int \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} \sec t \\ 0 \end{pmatrix} dt \\ &= \vec{X} \int \begin{pmatrix} -\sec t \sin t \\ \cos t \sec t \end{pmatrix} dt = \vec{X} \int \begin{pmatrix} -\frac{\sin t}{\cos t} \\ 1 \end{pmatrix} dt \end{aligned}$$

P4

u-substitution

$$u = \cos t$$

$$du = -\sin t dt$$

$$\vec{X}_p = \vec{X} \int \left( \frac{\frac{1}{u} du}{1 dt} \right) = \vec{X} \begin{pmatrix} \ln|\cos t| \\ t \end{pmatrix}$$

$$= \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} \begin{pmatrix} \ln|\cos t| \\ t \end{pmatrix}$$

$$= \begin{pmatrix} -\sin t \ln|\cos t| + t \cos t \\ \ln|\cos t| \cos t + t \sin t \end{pmatrix}$$

$$\vec{X} = C_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + C_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} t + \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \ln|\cos t|$$

$$\text{Ex 3} \quad \vec{X}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \vec{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

$$|A - rI| = 0$$

$$\begin{vmatrix} -r & 2 \\ -1 & 3-r \end{vmatrix} = -r(3-r) + 2 = 0$$

$$-3r + r^2 + 2 = 0$$

$$r^2 - 3r + 2 = 0$$

$$(r-2)(r-1) = 0 \quad r_1 = 2, r_2 = 1$$

$$(A - r_1 I) \vec{u} = \vec{0}$$

$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u_1 = u_2 \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_s$$

$$(A - r_2 I) \vec{u} = \vec{0}$$

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-u_1 + 2u_2 = 0$$

$$u_1 = 2u_2$$

$$\vec{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_s$$

$$\text{p5) } \vec{X}_c = C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\underline{X} = \begin{pmatrix} e^{2t} & 2e^t \\ e^{2t} & e^t \end{pmatrix}$$

$$\underline{X}^{-1} = \frac{1}{e^{3t} - 2e^{3t}} \begin{pmatrix} e^t & -2e^t \\ -e^{2t} & e^{2t} \end{pmatrix} = \begin{pmatrix} -e^{-2t} & 2e^{-2t} \\ e^{-t} & -e^{-t} \end{pmatrix}$$

$$\vec{X}_p = \underline{X} \int \underline{X}^{-1} \vec{f}$$

$$= \underline{X} \int \begin{pmatrix} -e^{-2t} & 2e^{-2t} \\ e^{-t} & -e^{-t} \end{pmatrix} \begin{pmatrix} e^t \\ -e^t \end{pmatrix} dt$$

$$= \underline{X} \int \begin{pmatrix} -e^{-t} & -2e^{-t} \\ 1 & 1 \end{pmatrix} dt$$

$$= \underline{X} \int \begin{pmatrix} -3e^{-t} \\ 2 \end{pmatrix} dt$$

$$= \begin{pmatrix} e^{2t} & 2e^t \\ e^{2t} & e^t \end{pmatrix} \begin{pmatrix} 3e^t \\ 2t \end{pmatrix} = \begin{pmatrix} 3e^{3t} + 4te^t \\ 3e^{3t} + 2te^t \end{pmatrix}$$

$$\vec{X} = C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix} + \begin{pmatrix} 4e^t t \\ 2e^t t \end{pmatrix}$$

Ex 4

$$\frac{dx}{dt} = 3x - 3y + 4$$

$$\frac{dy}{dt} = 2x - 2y - 1$$

p.6

$$A = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}$$

$$\vec{f} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = A \begin{pmatrix} x \\ y \end{pmatrix} + \vec{f}$$

$$\begin{vmatrix} 3-r & -3 \\ 2 & -2-r \end{vmatrix} = (3-r)(-2-r) + 6 = 0$$

$$-6 + 2r - 3r + r^2 + 6 = 0$$

$$r^2 - r = 0 \quad r_1 = 0 \quad r_2 = 1$$

$$(A - 0I)\vec{u} = \vec{0}$$

$$\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u_1 = u_2$$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_s$$

$$(A - I)\vec{u} = \vec{0}$$

$$\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2u_1 - 3u_2 = 0$$

$$2u_1 = 3u_2$$

$$u_1 = \frac{3}{2}u_2$$

$$\vec{u}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}_s$$

$$\vec{X}_c = C_1 e^{0t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 3 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \quad X^{-1} = \frac{1}{2e^t - 3e^t} \begin{pmatrix} 2e^t & -3e^t \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 3 \\ +1e^{-t} & -e^{-t} \end{pmatrix}$$

$$\vec{X}_p = X \int X^{-1} \vec{f} dt = X \int \begin{pmatrix} -2 & 3 \\ e^t & -e^{-t} \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} dt = X \int \begin{pmatrix} -8-3 \\ 4e^t + e^{-t} \end{pmatrix} dt$$

$$= X \int \begin{pmatrix} -11 \\ 5e^t \end{pmatrix} dt = \begin{pmatrix} 1 & 3e^t \\ 1 & 2e^t \end{pmatrix} \begin{pmatrix} -11t \\ -5e^{-t} \end{pmatrix} = \begin{pmatrix} -11t - 15 \\ -11t - 10 \end{pmatrix}$$

$$\vec{X} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -11t - 15 \\ -11t - 10 \end{pmatrix}$$