Absolute Maximum & Minimum Values

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

① Find the values of f at the critical points of f in D

② Find the extreme values of f on the boundary of D

③ Largest values from ① and ② is the absolute maximum value; smallest is the absolute minimum

P810 Find the absolute maximum and minimum values of f on the set D

29. $f(x,y) = x^4 + y^4 - 4xy + 2$

$D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$

→ First find all critical points of f on D

$f_x = 4x^3 - 4y = 0 \quad x^3 = y$

$f_y = 4y^3 - 4x = 0 \quad y^3 = x \rightarrow (y^3)^3 = y \quad y^9 - y = 0$

$y(y^8 - 1) = 0$

$y = 0, y = 1, y = -1$

If $y = 0 \rightarrow x^3 = 0 \rightarrow x = 0 \quad (0,0)$

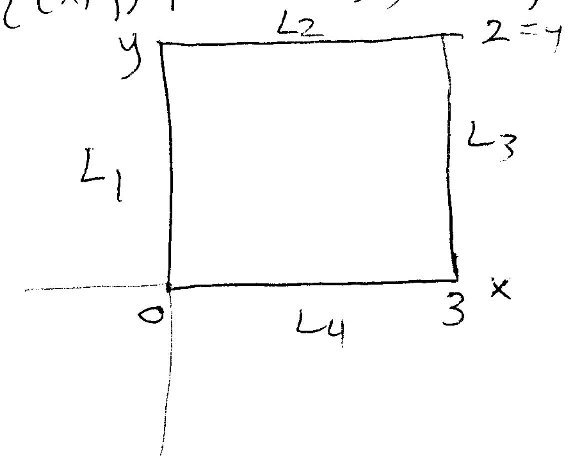
If $y = 1 \rightarrow x^3 = 1 \rightarrow x = 1 \quad (1,1)$

If $y = -1 \rightarrow x^3 = -1 \rightarrow x = -1 \quad (-1,-1)$

Our critical points are : $(0,0)$, $(1,1)$, $(-1,-1)$
but $(-1,-1)$ is not in D so we won't find $f(-1,-1)$.

$f(0,0) = 2$
$f(1,1) = 0$

→ Next draw D on the (xy)-plane
$D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$



→ We need to find the extreme values of f on L_1, L_2, L_3, L_4

L_1: L_1 is the line $x=0$ y between 0 and 2

$f(0,y) = 0^4 + y^4 - 4(0)y + 2 = y^4 + 2$

Extreme values of $f(0,y)$ happen at its end points and critical points.

Find $f_y(0,y) = 4y^3 = 0 \rightarrow y = 0$

$f(0,0) = 2$

At the end points : $f(0,0) = 2$
$f(0,2) = 0^4 + 2^4 - 4(0)(2) + 2$
$= 18$

L_2: On L_2 $y=2$ (y is fixed, x varies)

$$f(x,2) = x^4 + 2^4 - 4x(2) + 2$$
$$= x^4 - 8x + 18$$
$$f_x(x,2) = 4x^3 - 8 = 0$$
$$x^3 = 2 \rightarrow x = 2^{1/3}$$
$$f(2^{1/3},2) = (2^{1/3})^4 - 8(2^{1/3}) + 18 = 10.44$$

End points of L_2: $f(0,2) = 18$

$f(3,2) = 75$

L_3: On L_3 $x=3$ $f(3,y) = 3^4 + y^4 - 4(3)y + 2$

$$= 81 + y^4 - 12y + 2$$
$$= 83 + y^4 - 12y$$
$$f_y(3,y) = 4y^3 - 12 = 0$$
$$y^3 = 3 \quad y = 3^{1/3}$$

$$f(3,3^{1/3}) = 83 + (3^{1/3})^4 - 12(3^{1/3}) = 70.02$$

End points of L_3: $f(3,2) = 75$

$f(3,0) = 83$

L_4: On L_4 $y=0$

$$f(x,0) = x^4 + 0^4 - 4x(0) + 2 = x^4 + 2$$
$$f_x(x,0) = 4x^3 = 0 \quad x = 0$$
$$f(0,0) = 2$$
$$f(3,0) = 83$$

So we compare $f(0,0)=2$

$f(1,1)=0$

$f(0,2)=18$

$f(3,2)=75$

$f(3,0)=83$

$f(3,0)=83$ is our absolute max

$f(1,1)=0$ is our absolute min

p825 Find the absolute maximum and minimum values of f on the set D

55. $f(x,y)=4xy^2-x^2y^2-xy^3$; D is the closed triangular region in the xy-plane with vertices $(0,0)$, $(0,6)$, and $(6,0)$.

→ Find critical points of f on D

$f_x=4y^2-2xy^2-y^3=0 \rightarrow y^2(4-2x-y)=0$

$f_y=8xy-2x^2y-3xy^2=0 \rightarrow xy(8-2x-3y)=0$

$y^2(4-2x-y)=0$ if $y=0$ or if $y=4-2x$

If $y=0$ $xy(8-2x-3y)$ becomes 0 so (this is unusual) any point $(x,0)$ is a critical point. $f(x,0)=0$

If $y=4-2x$ then $xy(8-2x-3y)=x(4-2x)(8-2x-12+6x)$
$=x(4-2x)(-4+4x)=0$

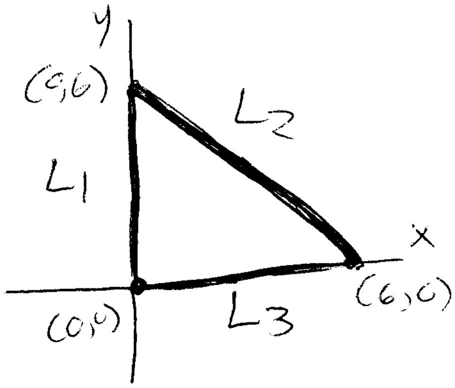
If $x=0$, if $x=2$, if $x=1$

$y=4-2(0)$ $=4$ $y=4-2(2)$ $=0$ $y=4-2$ $=2$

Critical points are $(x,0)$
$(0,4)$
$(2,0)$
$(1,2)$

$f(x,0)=0$
$f(0,4)=0$
$f(2,0)=0$
$f(1,2)=4(1)(2)^2-(1)^2(2)^2-(1)(2)^3$
$= 16 - 4 - 8$
$= 4$

Next we check the boundaries



L_1: on L_1 $x=0$ $f(0,y)=0$
We don't have any work to do here.

L_2: We need to describe L_2. $m=\frac{0-6}{6-0}=-1$
$y-6=-1(x-0)$ $y=-x+6$ describes L_2

$f(x,-x+6)=4x(6-x)^2-x^2(6-x)^2-x(6-x)^3$
$=(6-x)^2[4x-\cancel{x^2}-6x+\cancel{x^2}]$
$=-2x(6-x)^2=-2x(36-12x+x^2)$
$=-72x+24x^2-2x^3$

$f_x(x,6-x)=-72+48x-6x^2=0$
$6(-12+8x-x^2)=0$
$-6(x^2-8x+12)=0$

$$-6(x-6)(x-2)=0$$

$$x=6 \quad x=2$$

$x=6$: $f(6,-6+6)=f(6,0)=0$

$x=2$: $f(2,4)=4(2)(4)^2-(2)^2(4)^2-2(4)^3$

$$=-64$$

Check end points: $f(0,6)=0$

$f(6,0)=0$

L_3: on L_3 $y=0$ $f(x,0)=0$ No work to do.

Compare $f(x,0)=0$

$f(0,4)=0$

$f(1,2)=4$

$f(0,y)=0$

$f(6,0)=0$

$f(2,4)=-64$

$$\boxed{\begin{array}{l}\text{Absolute max } f(1,2)=4\\ \text{Absolute min } f(2,4)=-64\end{array}}$$

* Notice: There are many things I like about this problem (Extreme values aren't at the end points of L_1, L_2, L_3)

However the numbers here and the functions are not very pleasant. Try to understand the routine for these problems. The critical points won't be this difficult to find on your test.