Arc Length Practice

Def: the length of a curve $\vec{r}(t)$ where, $a \le t \le b$ is

$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

Find the length of the curve

Ex 1: $\vec{r}(t) = \langle 2t, t^2, \frac{1}{3}t^3 \rangle$

$0 \le t \le 1$

☆ 1st find $\vec{r}\,' = \langle 2, 2t, t^2 \rangle$

$$L = \int_0^1 \sqrt{2^2 + (2t)^2 + (t^2)^2}\,dt$$

$$= \int_0^1 \sqrt{4 + 4t^2 + t^4}\,dt$$

☆ After you simplify, if there are 3 terms left it probably factors as $(a+b)^2 = a^2 + 2ab + b^2$

$$\int_0^1 \sqrt{(t^2 + 2)^2}\, dt$$

I thought what would give me t^4 if I squared it

likewise what would give us 4

☆ Check to see if this works!

$(t^2)^2 + 2t^2(2) + 2^2 = t^4 + 4t^2 + 4$ ✓

Now we can take the sqrt

$$\int_0^1 t^2+2\, dt = \frac{1}{3}t^3+2t\, \Big|_0^1 = \boxed{\frac{1}{3}+2}$$

Ex 2: $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + \ln(\cos t)\,\hat{k}$

$0 \le t \le \frac{\pi}{4}$

☆ Find $\vec{r}\,'(t)$! $\vec{r}\,' = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \frac{(-\sin t)}{\cos t}\,\hat{k}$

$$L = \int_0^{\pi/4} \sqrt{(-\sin t)^2 + (\cos t)^2 + \left(\frac{-\sin t}{\cos t}\right)^2}\, dt$$

$$= \int_0^{\pi/4} \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{=1} + \frac{\sin^2 t}{\cos^2 t}}\, dt$$

$$L = \int_0^{\pi/4} \sqrt{\underbrace{1 + \tan^2 t}_{\sec^2 t}}\, dt$$

$\sin^2 t + \cos^2 t = 1$ so $\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$

or $\tan^2 t + 1 = \sec^2 t$

$$= \int_0^{\pi/4} \sqrt{\sec^2 t}\, dt = \int_0^{\pi/4} \sec t\, dt$$

oh boy! I don't expect you to remember this

$$= \int_0^{\pi/4} \sec t \underbrace{\left(\frac{\sec t + \tan t}{\sec t + \tan t}\right)}_{\text{multiplying by 1, tricky}} dt$$

$$= \int_0^{\pi/4} \frac{\sec^2 t + \sec t \tan t}{\sec t + \tan t}\, dt$$

$u = \sec t + \tan t$

$du = \sec t \tan t + \sec^2 t \;\; dt$

$u(0) = \sec 0 + \tan 0 = 1$

$u(\pi/4) = \sec \frac{\pi}{4} + \tan \frac{\pi}{4} = \sqrt{2} + 1$

$$\int_1^{\sqrt{2}+1} \frac{1}{u}\, du = \ln|u| \Big|_1^{\sqrt{2}+1} = \boxed{\ln(\sqrt{2}+1)} - \cancel{\ln 1}$$

Ex 3:

$$\vec{r}(t) = 12t\,\hat{i} + 8t^{3/2}\,\hat{j} + 3t^2\,k$$

$$0 \le t \le 1$$

$$\vec{r}\,' = 12\hat{i} + 12t^{1/2}\hat{j} + 6t\hat{k}$$

$$L = \int_0^1 |\vec{r}\,'|\,dt = \int_0^1 \sqrt{144 + (12t^{1/2})^2 + (6t)^2}\,dt$$

$$= \int_0^1 \sqrt{144 + 144t + 36t^2}\,dt$$

☆ Check to see if it factors

$$= \int_0^1 \sqrt{(6t + 12)^2}\,dt$$

$$36t^2 + 2(6t)(12) + 144 \quad ✓$$

$$= \int_0^1 6t + 12\,dt$$

$$= 3t^2 + 12t \Big|_0^1 = 3 + 12 = \boxed{15}$$

Ex 4: $\vec{r}(t) = \langle e^t \sin t, e^t \cos t, e^t \rangle$

for $0 \le t \le \ln 2$

$$\vec{r}\,'(t) = \langle \underbrace{e^t \sin t + e^t \cos t}_{\text{product rule}}, \underbrace{e^t \cos t - e^t \sin t}_{\text{product rule}}, e^t \rangle$$

$$L = \int_0^{\ln 2} |\vec{r}\,'(t)|\, dt$$

$$= \int_0^{\ln 2} \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2 + (e^t)^2}\, dt$$

$$= \int_0^{\ln 2} \sqrt{e^{2t} \sin^2 t + \cancel{2 e^{2t} \sin t \cos t} + e^{2t} \cos^2 t + e^{2t} \cos^2 t - \cancel{2 e^{2t} \cos t \sin t} + e^{2t} \sin^2 t + e^{2t}}\, dt$$

☆ Recall $(e^t)^2 = e^{2t}$

$$= \int_0^{\ln 2} \sqrt{e^{2t} \underbrace{(\sin^2 t + \cos^2 t)}_{=1} + e^{2t} \underbrace{(\sin^2 t + \cos^2 t)}_{=1} + e^{2t}}\, dt$$

$$= \int_0^{\ln 2} \sqrt{3 \underbrace{e^{2t}}_{(e^t)^2}}\, dt = \int_0^{\ln 2} \sqrt{3}\, e^t\, dt$$

$$= \sqrt{3}\, e^t \Big|_0^{\ln 2} = \sqrt{3}(e^{\ln 2} - e^0) = \sqrt{3}(2-1) = \boxed{\sqrt{3}}$$

Ex 5: $\vec{r}(t) = \left\langle t+4,\ \frac{1}{4}t^2 - \frac{1}{2}\ln t,\ 12 \right\rangle$

$1 \le t \le 2$

$\vec{r}\,' = \left\langle 1,\ \frac{1}{2}t - \frac{1}{2t},\ 0 \right\rangle$

$$L = \int_1^2 \sqrt{1^2 + \left(\frac{1}{2}t - \frac{1}{2t}\right)^2 + 0^2}\, dt$$

$$= \int_1^2 \sqrt{1 + \frac{1}{4}t^2 - \cancel{2}\left(\frac{1}{\cancel{2}}\cancel{t}\right)\left(\frac{1}{2\cancel{t}}\right) + \frac{1}{4t^2}}\, dt$$

$$= \int_1^2 \sqrt{1 + \frac{1}{4}t^2 - \frac{1}{2} + \frac{1}{4t^2}}\, dt$$

$$= \int_1^2 \sqrt{\underbrace{\frac{1}{2} + \frac{1}{4}t^2 + \frac{1}{4t^2}}_{\text{simplifies to 3 terms - factor}}}\, dt$$

$$= \int_1^2 \sqrt{\left(\frac{1}{2}t + \frac{1}{2t}\right)^2}\, dt$$

☆ Check $\frac{1}{4}t^2 + 2\left(\frac{1}{\cancel{2}}\cancel{t}\right)\left(\frac{1}{2\cancel{t}}\right) + \frac{1}{4t^2}$ ✓

$$= \int_1^2 \frac{1}{2}t + \frac{1}{2t}\, dt = \frac{1}{4}t^2 + \frac{1}{2}\ln t \Big|_1^2$$

$$= \frac{4}{4} + \frac{1}{2}\ln 2 - \left(\frac{1}{4} + \cancel{\frac{1}{2}\ln 1}\right) = \boxed{\frac{3}{4} + \frac{1}{2}\ln 2}$$