Test 4 Review Problems mostly from Calculus by Edwards & Penney

1. $m = \int_C \sigma(x,y)\,ds \qquad \sigma = x^2+y^2$

$= \int_C (x^2+y^2)\,ds \qquad \vec{r} = \langle \underset{x}{4t-1}, \underset{y}{3t+1} \rangle \qquad -1 \le t \le 1$

$$= \int_{-1}^{1} \left[\underset{x}{(4t-1)^2} + \underset{y}{(3t+1)^2} \right] \underbrace{\|\vec{r}\,'(t)\|\,dt}_{ds}$$

$$= \int_{-1}^{1} \left(16t^2 - 8t + 1 + 9t^2 + 6t + 1\right) \sqrt{\underset{\frac{dx}{dt}}{4}^2 + \underset{\frac{dy}{dt}}{3}^2}\,dt$$

$$= \int_{-1}^{1} \left(25t^2 - 2t + 2\right)\sqrt{25}\,dt$$

$$= 5\int_{-1}^{1} \underbrace{25t^2 + 2\,dt}_{\text{even function}} + \cancel{5\int_{-1}^{1} -2t\,dt} \quad 0 \quad \text{odd}$$

$$= 5 \cdot 2 \int_0^1 25t^2 + 2\,dt = \boxed{10\left[\frac{25}{3} + 2\right]}$$

2. line integral wrt arc length $= \int_C f(x,y,z)\,ds$

$f = 2x - y$ where C is the circle $x^2+y^2 = 25$ in the plane $z = 4$.

Parametric rep:

$x = 5\cos t$
$y = 5\sin t$
$z = 4$
$0 \le t \le 2\pi$

$\vec{r} = \langle 5\cos t, 5\sin t, 4 \rangle$

$\vec{r}\,' = \langle -5\sin t, 5\cos t, 0 \rangle$

ds = 5 dt for problems 1 & 2 this is not always going to happen

$$\int_C (2x-y)\,ds = \int_0^{2\pi} (2\cdot 5\cos t - 5\sin t)\sqrt{25\sin^2 t + 25\cos^2 t}\,dt$$

$$= \int_0^{2\pi} (10\cos t - 5\sin t)\,5\,dt = 5\left(10\sin t + 5\cos t\right)\Big|_0^{2\pi} = \boxed{0}$$

★ More exciting problem: C is the half circle $x = \sqrt{25-y^2}$ in the plane $z = 4$ $\longrightarrow$ $x = 5\cos t$, $y = 5\sin t$, $z = 4$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, $f = 2x - y$ $\boxed{\text{Ans: } 100}$

3. $W = \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$

$\vec{F} = \langle x^2 y, xy^3 \rangle$

C_1: line segment $(-1,1)$ to $(2,1)$

C_2: then line segment $(2,1)$ to $(2,5)$

C_1: line segment from $\vec{r}_0$ to $\vec{r}_1$ $\quad \vec{r} = (1-t)\vec{r}_0 + \vec{r}_1 t \quad 0 \le t \le 1$

$$\vec{r} = (1-t)\langle -1,1\rangle + \langle 2,1\rangle t$$

$$= \langle -1+t, 1-t\rangle + \langle 2t, t\rangle = \langle \underset{=x}{-1+3t}, \underset{=y}{1} \rangle$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \langle x^2 y, xy^3 \rangle \cdot d\vec{r} = \int_0^1 \langle (-1+3t)^2 \cdot 1, (-1+3t) \cdot 1^3 \rangle \cdot \underbrace{\langle 3, 0 \rangle dt}$$

$d\vec{r} = \vec{r}\,' dt$

$$= \int_0^1 3(-1+3t)^2 dt = \int_0^1 3(1-6t+9t^2)\,dt$$

$$= 3\left[t - 3t^2 + 3t^3\right]_0^1 = \boxed{3}$$

-not final ans

C_2: line segment $\vec{r} = (1-t)\vec{r}_0 + \vec{r}_1 t \quad 0 \le t \le 1$

$$\vec{r} = \langle 2,1\rangle(1-t) + \langle 2,5\rangle t = \langle 2-2t, 1-t\rangle + \langle 2t, 5t\rangle$$

$$= \langle 2, 1+4t \rangle \quad 0 \le t \le 1$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \langle 2^2 \cdot (1+4t), 2 \cdot (1+4t)^3 \rangle \cdot \underbrace{\langle 0, 4 \rangle dt}_{\vec{r}\,' dt}$$

$\langle x^2 y, xy^3 \rangle$

$$= \int_0^1 2 \cdot \underline{4}(1+4t)^3 \underline{dt} = \int_1^5 2u^3 du = \frac{2}{4}u^4 \Big|_1^5$$

$u = 1+4t \quad du = 4dt$

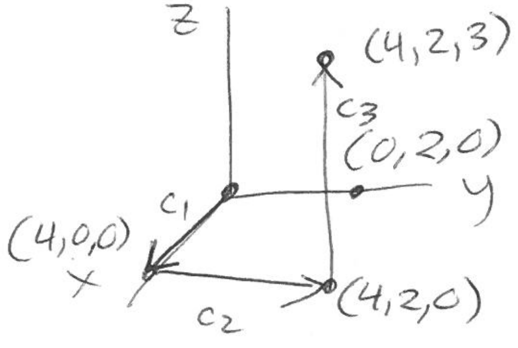
$u(0) = 1 \quad u(1) = 5$

$$= \boxed{\frac{1}{2}\left[5^4 - 1^4\right]}$$

Ans: $\boxed{3 + \frac{1}{2}\left[5^4 - 1\right]} = 3 + \frac{1}{2}\left[625 - 1\right] = 3 + 312 = 315$

4. $\int_C \vec{F} \cdot d\vec{r} = \int_C \langle 2x+3y, 3x+2y, 3z^2 \rangle \cdot d\vec{r}$

C is the path from (0,0,0) to (4,2,3) along 3 line segments // to the x, y, & z axes in that order



We could do 3 reps for line segments & add them together, but instead we are going to hopefully take a shortcut.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y & 3x+2y & 3z^2 \end{vmatrix}$$

$= \langle 0-0, -(0-0), 3-3 \rangle = \vec{0} \rightarrow \vec{F}$ is conservative but conservative vectors are path independent. This means instead of going along 3 line segments that work will be equal to the work going directly from (0,0,0) to (4,2,3). We can either find that line segment or find f & use the Fundamental thm. I'm going to find f. $\vec{F}$ is conservative so

$\vec{F} = \nabla f$

$f_x = 2x + 3y \quad f_y = 3x + 2y \quad f_z = 3z^2$

$f = x^2 + 3xy + g(y,z)$

$f_y = 0 + \underline{3x} + g_y(y,z) = \underline{3x} + 2y$

$g_y(y,z) = 2y$

$g(y,z) = y^2 + h(z)$

$f = x^2 + 3xy + y^2 + h(z)$

$f_z = 0 + 0 + 0 + h'(z) = 3z^2$

$h(z) = z^3 + k$

$f = x^2 + 3xy + y^2 + z^3$

$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$= f(4,2,3) - f(0,0,0) = 4^2 + 3 \cdot 4 \cdot 2 + 2^2 + 3^3 - 0$

$= \boxed{16 + 24 + 4 + 27} = 71$

5. $\vec{F} = \langle x + \arctan y, \frac{x+y}{1+y^2} \rangle = \langle \underset{P}{\underset{\parallel}{x + \tan^{-1} y}}, \underset{Q}{\underset{\parallel}{\frac{x}{1+y^2} + \frac{y}{1+y^2}}} \rangle$

$$\frac{\partial P}{\partial y} = \frac{1}{1+y^2} = \frac{\partial Q}{\partial x} = \frac{1}{1+y^2} \quad \checkmark$$

Alternatively, you could do $\text{curl}\,\vec{F} = \vec{0}$.

$f_x = x + \tan^{-1} y \qquad f_y = \frac{x}{1+y^2} + \frac{y}{1+y^2}$

$f = \frac{1}{2}x^2 + x\tan^{-1} y + g(y)$

$$f_y = 0 + \underline{\frac{x}{1+y^2}} + g'(y) = \underline{\frac{x}{1+y^2}} + \frac{y}{1+y^2}$$

$$g'(y) = \frac{y}{1+y^2}$$

$$g(y) = \int \frac{y}{1+y^2}\,dy = \int \frac{1}{2}\frac{1}{u}\,du$$

$u = 1+y^2$

$du = 2y\,dy$

$\frac{1}{2}du = y\,dy$

$$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1+y^2) + C$$

$$\boxed{f = \frac{1}{2}x^2 + x\tan^{-1} y + \frac{1}{2}\ln(1+y^2) + C}$$

6. $\vec{F} = \langle \underbrace{e^x \sin y + \tan y}_{P}, \underbrace{e^x \cos y + x\sec^2 y + \frac{3}{1+y}}_{Q} \rangle$

$$\frac{\partial P}{\partial y} = e^x \cos y + \sec^2 y = \frac{\partial Q}{\partial x} = e^x \cos y + 1\sec^2 y + 0 \checkmark$$

or do $\text{curl}\vec{F} = \vec{0}$

Conservative.

$f_x = e^x \sin y + \tan y \qquad f_y = e^x \cos y + x\sec^2 y + \frac{3}{1+y}$

$f = e^x \sin y + x\tan y + g(y)$

$f_y = \underline{e^x \cos y} + x\sec^2 y + g'(y) = \underline{e^x \cos y} + x\sec^2 y + \frac{3}{1+y}$

$g'(y) = \frac{3}{1+y}$

$g(y) = 3\ln|1+y| + k$

$$\boxed{f = e^x \sin y + x\tan y + 3\ln|1+y| + k}$$

7. $\vec{F} = \langle -\sin(x+2z) + \frac{1}{\sqrt{x}},\ 3ze^{3yz},\ -2\sin(x+2z) + 3ye^{3yz} + z^2 \rangle$

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\sin(x+2z) + \frac{1}{\sqrt{x}} & 3ze^{3yz} & -2\sin(x+2z) + 3ye^{3yz} + z^2 \end{vmatrix}$$

$$= \langle 0 + 3e^{3yz} + 9yze^{3yz} + 0 - (3e^{3yz} + 9zye^{3yz}),\ -(-2\cos(x+2z) - (-2\cos(x+2z))),\ 0-0 \rangle$$

$= \vec{0} \rightarrow$ conservative $\rightarrow$ path independent

$f_x = -\sin(x+2z) + x^{-1/2}$ $\quad f_y = 3ze^{3yz}$ $\quad f_z = -2\sin(x+2z) + 3ye^{3yz} + z^2$

$f = \cos(x+2z) + 2x^{1/2} + g(y,z)$

$f_y = 0 + 0 + g_y(y,z) = 3ze^{3yz}$

$g(y,z) = e^{3yz} + h(z)$

more detail if you need it $\int 3ze^{3yz}\,dy = \int e^u\,du = e^u = e^{3yz}$

$u = 3yz$

$du = \frac{\partial}{\partial y}(3yz) = 3z\,dy$

$f = \cos(x+2z) + 2x^{1/2} + e^{3yz} + h(z)$

$f_z = \quad -2\sin(x+2z) + 0 + 3ye^{3yz} + h'(z) = -2\sin(x+2z) + 3ye^{3yz} + z^2$

$h(z) = \frac{1}{3}z^3 + K$

$f = \cos(x+2z) + 2\sqrt{x} + e^{3yz} + \frac{1}{3}z^3 + K$

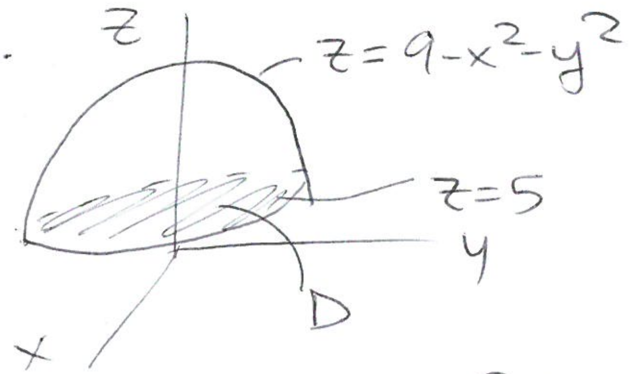
$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$= f(1,1,1) - f(4,0,-2) = \cos 3 + 2 + e^3 + \frac{1}{3} - [\overset{1}{\cancel{\cos 0}} + 2 \cdot 2 + \overset{1}{\cancel{e^0}} - \frac{8}{3}]$

$= \boxed{\cos(3) + e^3 - 1}$

☆ This problem was a little bit too much. Remember if we don't have 2 points we need to find them by calculating $\vec{r}(b)$ & $\vec{r}(a)$

8.



Surface Area $= \iint_S 1\,dS = \iint_D \|\vec{r}_u \times \vec{r}_v\|\,du\,dv$

Since ~~xxx~~ we have $z = f(x,y)$ we can use our shortcut $\|\vec{r}_x \times \vec{r}_y\| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$

$$\iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\;dA$$

$$= \iint_D \sqrt{4x^2 + 4y^2 + 1}\;dA = \int_0^{2\pi}\int_0^2 \sqrt{4r^2+1}\;r\,dr\,d\theta$$

$u = 4r^2 + 1$

$du = 8r\,dr$

$\frac{1}{8}du = r\,dr$

D is circular so switching to polar

$$= \frac{2\pi}{8}\int_1^{17} \sqrt{u}\,du$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^{17}$$

$$= \boxed{\frac{\pi}{6}\left(17^{3/2} - 1\right)}$$

top = bottom

$9 - x^2 - y^2 = 5$

$9 - r^2 = 5$

$4 = r^2$

surface

9. $\boxed{z=xy}$ $x^2+y^2=36$

$$S.A. = \iint_D \|\vec{r}_x \times \vec{r}_y\| \, dA = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1} \, dA$$

$$= \iint_D \sqrt{y^2+x^2+1} \, dA = \int_0^{2\pi}\int_0^6 \sqrt{r^2+1} \, r\,dr\,d\theta$$

$x^2+y^2=36$

$u=r^2+1$

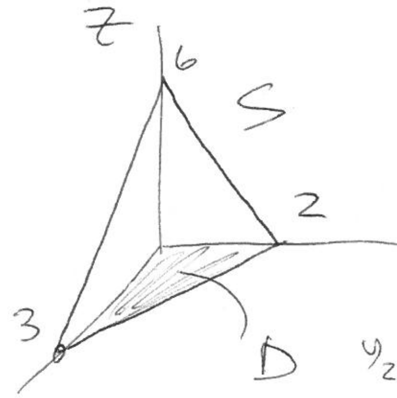
$du=2r\,dr$

$\frac{1}{2}du=r\,dr$

$$= 2\pi \cdot \frac{1}{2}\int_1^{37} \sqrt{u}\,du = \pi \, \frac{2}{3} u^{3/2}\Big|_1^{37}$$

$$= \frac{2\pi}{3}\left[37^{3/2}-1\right]$$

10. $\iint_S f\,dS$ $f=xyz$



need the equation of the plane containing those 3 points

$z=ax+by+c$

$(0,0,6)$: $6=a\cdot 0+b\cdot 0+c$

$0=6-2x-3y$

$y=2-\frac{2}{3}x$

$z=ax+by+6$

$(3,0,0)$: $0=3a+b\cdot 0+6$ $a=-2$ $z=6-2x+by$

$(0,2,0)$: $0=6+2b$ $b=-3$ $z=6-2x-3y$

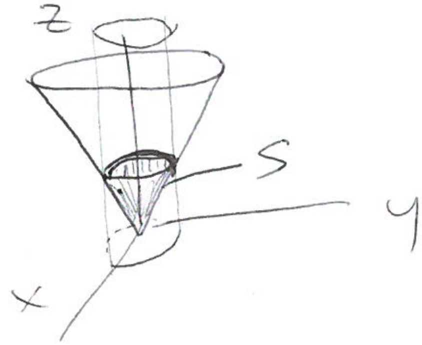
$$dS=\|\vec{r}_u\times\vec{r}_v\|\,du\,dv = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA = \sqrt{(-2)^2+(-3)^2+1}\,dA$$

$$\iint_S f\,dS = \iint_S xyz\,dS = \iint_D xy(6-2x-3y)\sqrt{14}\,dA$$

$$= \boxed{\int_0^3\int_0^{2-\frac{2}{3}x} 6xy-2x^2y-3xy^2\,dy\,dx} = \int_0^3 3xy^2-x^2y^2-xy^3\Big|_0^{2-\frac{2}{3}x}dx$$

$$=\int_0^3 3x\left(2-\tfrac{2}{3}x\right)^2-x^2\left(2-\tfrac{2}{3}x\right)^2-x\left(2-\tfrac{2}{3}x\right)^3dx = 9/5$$ (I didn't actually integrate this)

11. $\iint_S f\,dS = \iint_S z^2\,dS$



surface is the cone
$z = \sqrt{x^2+y^2}$

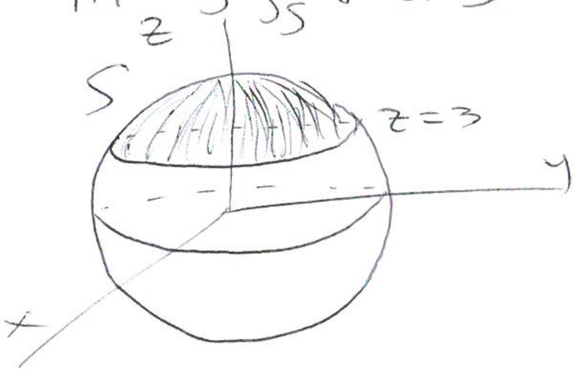
rep: $x=x,\ y=y,\ z=\sqrt{x^2+y^2}$

$$\iint_S z^2\,dS = \iint_D \left(\sqrt{x^2+y^2}\right)^2 \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\ dA$$

$$= \iint_D (x^2+y^2)\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}\ dA$$

$$= \int_0^{2\pi}\int_0^2 r^2\sqrt{2}\,r\,dr\,d\theta = 2\pi\sqrt{2}\,\frac{1}{4}r^4\Big|_0^2 = \boxed{8\pi\sqrt{2}}$$

12. $m = \iint_S \sigma\,dS = \iint_S (x^2+y^2)\,dS$



$z=\sqrt{25-x^2-y^2}$
$x=x$
$y=y$

$$\iint (x^2+y^2)\sqrt{\left(\frac{-x}{\sqrt{25-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{25-x^2-y^2}}\right)^2+1}\ dA$$

$$= \iint_D (x^2+y^2)\sqrt{\frac{x^2+y^2}{25-x^2-y^2}+\frac{25-x^2-y^2}{25-x^2-y^2}}\ dA = \iint_D (x^2+y^2)\frac{5}{\sqrt{25-x^2-y^2}}\,dA$$

$$= \int_0^{2\pi}\int_0^4 r^2\frac{5}{\sqrt{25-r^2}}\,\underline{r\,dr}\,d\theta$$

$u=25-r^2 \quad du=-2r\,dr$
$r^2=25-u \quad -\frac{1}{2}du = r\,dr$

top = bottom
$\sqrt{25-x^2-y^2}=3$
$25-r^2=9 \quad 16=r^2$

$\cancel{2}\pi\left(-\frac{1}{\cancel{2}}\right)5\int_{25}^{9}(25-u)u^{-1/2}\,du$

$= 5\pi\int_9^{25} 25u^{-1/2}-u^{1/2}\,du = 5\pi\left(50\sqrt{u}-\frac{2}{3}u^{3/2}\right)_9^{25}$

$= 5\pi\left(50(5)-\frac{2}{3}\cdot 5^3-\left(50(3)-\frac{2}{3}\cdot 27\right)\right)$