# Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms

① If $\sum b_n$ is convergent **AND** $a_n \le b_n$ for all $n$, then $\sum a_n$ is also convergent.

② If $\sum b_n$ is divergent **AND** $a_n \ge b_n$ for all $n$, then $\sum a_n$ is also divergent.

Examples from *Calculus: Early Transcendental Functions* By Larson, Hostetler, Edwards

$\boxed{\text{Ex 1}}$ $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ * We want to compare this to a p-series or a geometric series.

If we remove something positive from the denominator the fraction gets bigger.

$\left( \frac{1}{2+1} \le \frac{1}{2} \right)$ *

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \le \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{p-series } p=2>1 \text{ converges}$$

So $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the Comparison Test

Note: $\sum_{n=1}^{\infty} \frac{1}{n^2+1} \le \sum_{n=1}^{\infty} \frac{1}{1}$ but $\sum_{n=1}^{\infty} 1$ diverges so this isn't helpful. Being less than something that diverges doesn't mean that the series converges or diverges.

$\boxed{\text{Ex2}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+2} \leq \sum_{n=1}^{\infty} \frac{1}{3n^2}$ p-series $p=2>1$ converges

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ Converges by comparison test

$\boxed{\text{Ex 3}}$ $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n-2}$ * If we remove something negative from the denominator the fraction gets smaller $\left(\frac{1}{2-1}=1 \quad \frac{1}{2-1} \geq \frac{1}{2}\right)$. If you get confused try using numbers to figure out the inequality *

$$\sum_{n=3}^{\infty} \frac{1}{n-2} \geq \sum_{n=2}^{\infty} \frac{1}{n}$$ Harmonic series diverges

So $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n-2}$ diverges by the comparison test

$\boxed{\text{Ex4}}$ $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+1} \leq \sum_{n=0}^{\infty} \frac{1}{3^n} = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots$ Geometric series $r=\frac{1}{3}$ $\left|\frac{1}{3}\right|<1$ converges

So $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+1}$ converges by the comparison test.

$\boxed{\text{Ex5}}$ $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n+5} \leq \sum_{n=0}^{\infty} \frac{2^n}{3^n} = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots$

Geometric series $r=\frac{2}{3}$ $\left|\frac{2}{3}\right|<1$ converges

So $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n+5}$ converges by the comparison test

$\boxed{\text{Ex 6}}$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} \le \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

p-series $p=\frac{3}{2}>1$
Converges

So converges by the comparison test

$\boxed{\text{Ex 7}}$ $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}}$ * I can't remove the one because $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}} \ge \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3}}$ Being bigger than something that converges doesn't tell us anything *

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}} \le \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-\frac{1}{2}n^3}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{\frac{1}{2}n^3}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{\frac{1}{2}}\, n^{3/2}}$$

p-series $p=3/2>1$ converges

So $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}}$ converges by the comparison test.

* I wanted to get to $\frac{1}{\sqrt{an^3}}$ so I knew that I had to subtract something bigger than 1 that was a multiple of $n^3$. $\frac{1}{\sqrt{n^3-n^3}}$ obviously doesn't work so I tried $\frac{1}{2}n^3$

$n=2 \quad \frac{1}{2}n^3 = \frac{8}{4} = 2 > 1$

$n=3 \quad \frac{1}{2}n^3 = \frac{27}{2} > 1$

$\vdots$

$\boxed{\text{Ex 8}}$ $\sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} \geq \sum_{n=1}^{\infty} \frac{4^n}{3^n} = \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3 + \cdots$

Geometric series $r = \frac{4}{3}$ $\left|\frac{4}{3}\right| > 1$

diverges

so $\sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}}$ diverges by Comparison test

$\boxed{\text{Ex 9}}$ $\sum_{n=1}^{\infty} \frac{1}{3^n+2} \leq \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots$

Geometric series $r = \frac{1}{3}$ $\left|\frac{1}{3}\right| < 1$

Converges by Comparison test