Graphing Surfaces

* Being able to correctly graph surfaces is crucial for understanding the problems in Chapters 12 & 13

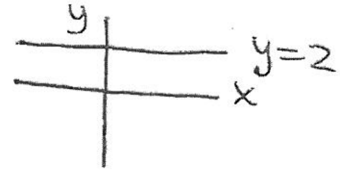
I expect you to be able to graph planes, spheres, ellipsoids, cones, paraboloids, cylinders (circular & noncircular)

Planes

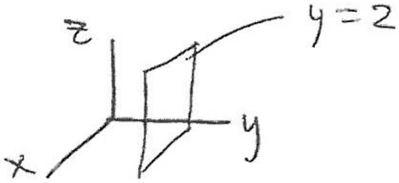
A plane is of the form $ax+by+cz+d=0$ (we've also written this as $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ & $z=ax+by+c$, assuming we can solve for z)

— Vertical planes: recall that a line in $\mathbb{R}^2$ (the xy-plane) is a plane in $\mathbb{R}^3$

$\boxed{\text{Ex 1}}$: $y=2$ in $\mathbb{R}^2$

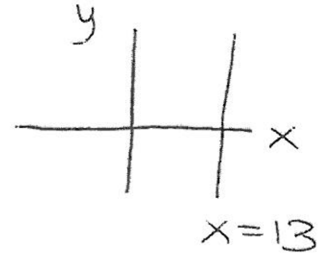


in $\mathbb{R}^3$



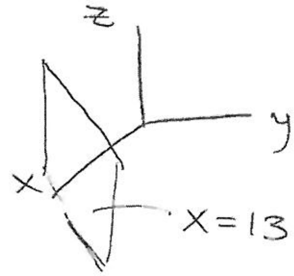
Notice: In both cases, y is always 2. In $\mathbb{R}^2$ for every value of x, $y=2$. In $\mathbb{R}^3$ for every value of x & z $y=2$.

$\boxed{\text{Ex 2}}$: $x = 13$ in $\mathbb{R}^2$



Note: Every value of y has $x = 13$

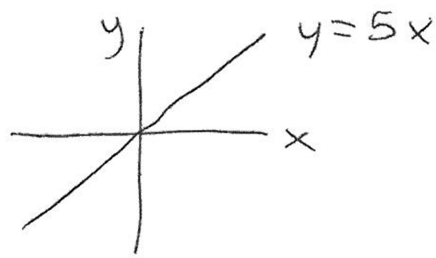
in $\mathbb{R}^3$



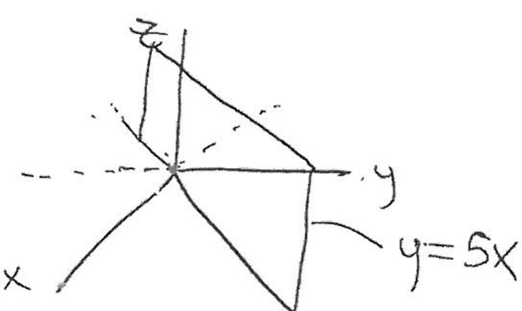
Every value of y & z has $x = 13$.

$\boxed{\text{Ex 3}}$: $y = 5x$ This kind of vertical plane can be kind of a hassle to draw. Usually this will end up being part of D.

$y = 5x$ in $\mathbb{R}^2$



in $\mathbb{R}^3$



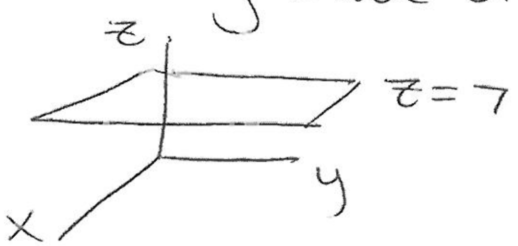
Draw the line $y = 5x$ in the xy-plane & then make it vertical. For every value of z (every horizontal cross-section) we get the line $y = 5x$

– Horizontal planes

$\boxed{Ex\,4}$: $z = 7$

It doesn't make sense to graph this in the xy-plane since we know that the xy-plane happens when $z = 0$.

- Notice that since we only have z for every value of x and y $z = 7$.



$z = 7$ is 7 units above the xy-plane, likewise if we had $z = -3$ it would be 3 units below the xy-plane.

– Tilted planes

- Tilted planes are of the form $z = ax + by + c$

To graph these usually we just want to know where the plane intersects the coordinate axes.

We know it intersects the x-axis when y & z are both zero

It intersects the y-axis when x & z are zero & it intersects the z-axis when x & y are zero.

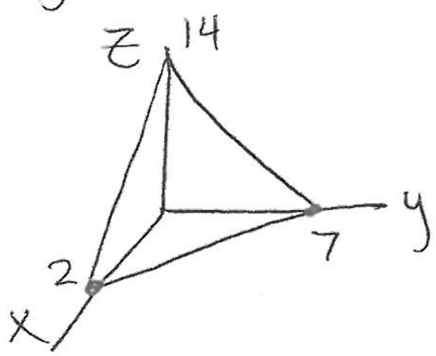
$\boxed{\text{Ex 5}}$: $z = 14 - 7x - 2y$

$x=0, y=0, z = 14 - 7\cdot 0 - 2\cdot 0 = 14$

$x=0, z=0, \quad 0 = 14 - 7\cdot 0 - 2y \quad 0 = 14 - 2y$

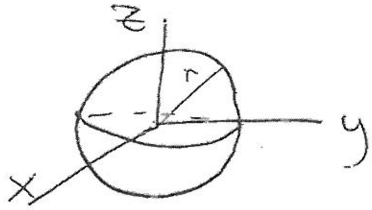
$y = 7$

$y=0, z=0 \quad 0 = 14 - 7x - 2\cdot 0 \rightarrow x = 2$



Spheres $(x-h)^2+(y-k)^2+(z-\ell)^2=r^2$
sphere centered at (h,k,ℓ) with radius r.

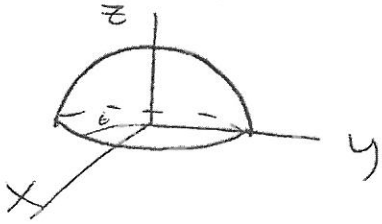
Lucky for us our spheres are usually centered at the origin: $x^2+y^2+z^2=r^2$



- Hemispheres

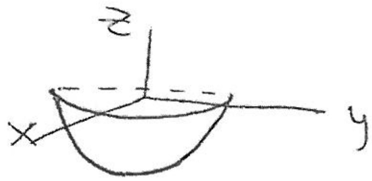
$\boxed{\text{Ex 6}}$: $z=\sqrt{36-x^2-y^2}$

Note! $z \geq 0$, this is the top half of the sphere. If we rewrote it we'd get $z^2=36-x^2-y^2$ or $x^2+y^2+z^2=36$ so it is centered at the origin with $r=6$



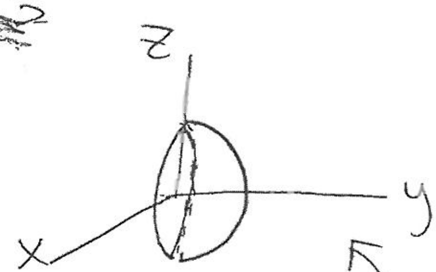
$\boxed{\text{Ex 7}}$: $z=-\sqrt{25-x^2-y^2}$

$z \le 0$, $r=5$



$\boxed{\text{Ex 8}}$: $y=\sqrt{9-x^2-z^2}$

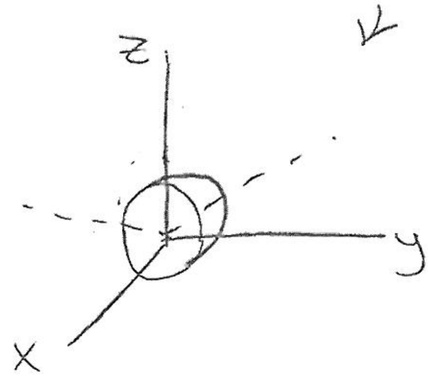
$y \ge 0$, $r=3$



Hard to draw

$\boxed{\text{Ex 9}}$: $x=-\sqrt{4-z^2-y^2}$

$x \le 0$, $r=2$



<u>Ellipsoids</u> $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

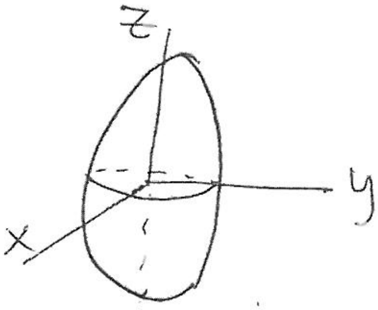
$a=b=c$ then we have a sphere
ellipsoids are basically smooshed spheres

Imagine taking an ellipse & spinning it around either the x-axis, y-axis or z-axis, the egg-shaped solid you get is an ellipsoid.

$\boxed{\text{Ex } 10}$: $2x^2+2y^2+z^2=18$

Note! $z=0 \rightarrow 2x^2+2y^2=18 \rightarrow x^2+y^2=9$

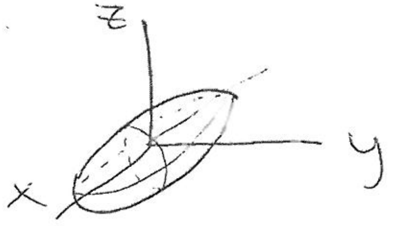
For fixed z we get circular cross-sections



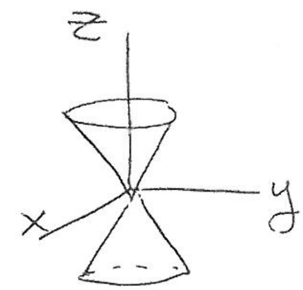
Remember the idea of level curves, if we fix z then we can determine what a horizontal cross-section looks like.

$\boxed{\text{Ex } 11}$: $3x^2+4y^2+4z^2=87$

For fixed x we get circles, so our cross-sections along the x-axis are circles



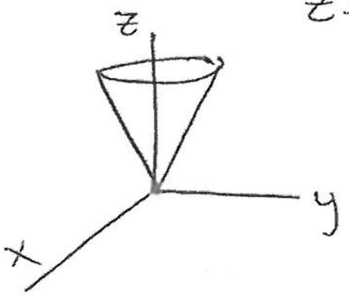
<u>Cones</u> $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



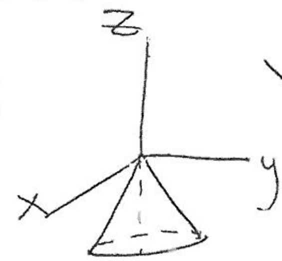
* If it is solved for z then we only get part of the cone.

$\boxed{\text{Ex 12}}$: $z = \sqrt{4x^2 + 4y^2}$

$z \geq 0$, top portion of cone



($z = -\sqrt{4x^2+4y^2}$

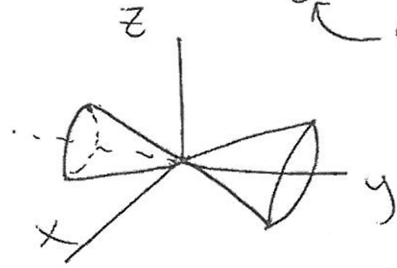


)

$\boxed{\text{Ex 13}}$: $y^2 = 2x^2 + 3z^2$.

centered on y-axis

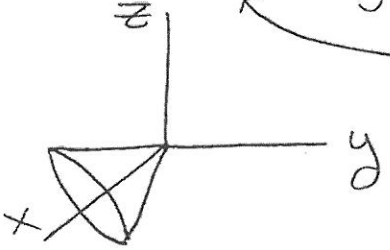
<u>Note</u>! For fixed y, cross-sections along y-axis are ellipses



$\boxed{\text{Ex 14}}$: $x = \sqrt{y^2 + z^2}$

centered on x-axis

$x \geq 0$



<u>Paraboloids</u> $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

axis it's centered around

Paraboloids are bowl shaped surfaces.
Don't confuse them with cones!

Cone: $z = \sqrt{x^2+y^2}$ or $z^2 = x^2+y^2$

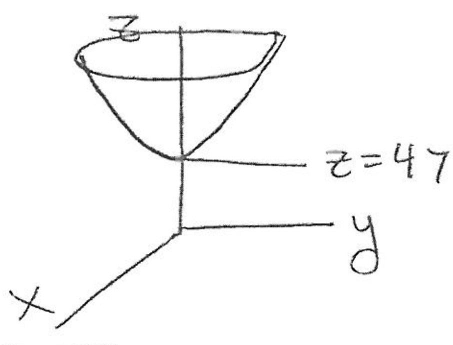
paraboloid $z = x^2+y^2$

$\boxed{\text{Ex 15}}$: $z = 47 + 3x^2 + 3y^2$

$x=0, y=0, z=47$

positive → opens up

axis it is centered on

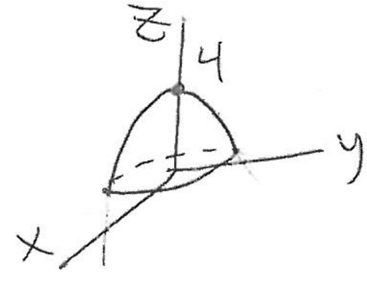


$\boxed{\text{Ex 16}}$: $z = 4 - x^2 - y^2$

negative opens down

Centered about z-axis

$x=0, y=0, z=4$



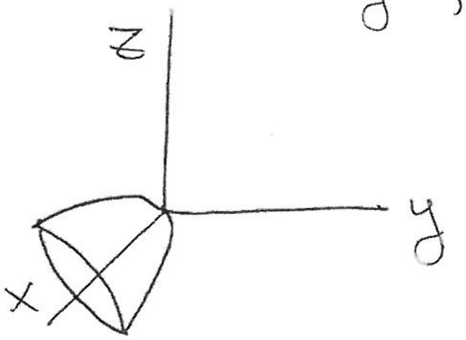
$\boxed{\text{Ex 17}}$: $x = 5y^2 + 7z^2$

opens to front

positive, different numbers mean cross sections ellipses

axis centered on

note $y=0, z=0 \to x=0$

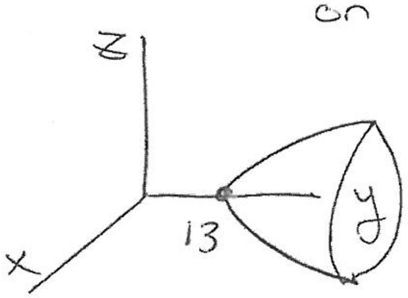


$\boxed{\text{Ex 18}}$: $y = 13 + 8x^2 + 8z^2$

positive, open to the right

axis centered on

$x=0, z=0, y=13$



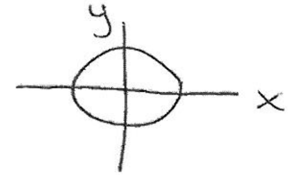
<u>Note</u>: Paraboloids are the shaped formed when parabolas are rotated about an axis. Also the sign of the variables it is not centered about must be the same (both positive or both negative

Cylinders

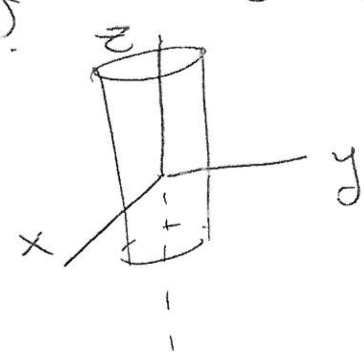
- Circular cylinders: we know that a line in 2D is a plane in 3D, likewise a circle in 2D is a cylinder in 3D.

$\boxed{\text{Ex 19}}$: $x^2+y^2=9$ in $\mathbb{R}^2$

$r=3$



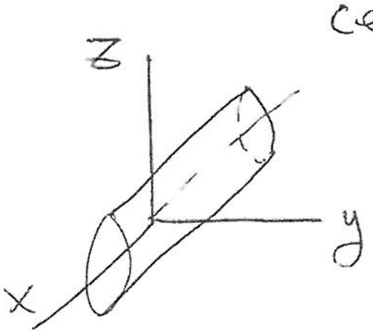
Note: Since z isn't in the equation for every value of z we get the same result, $x^2+y^2=9$. Every cross-section along the z-axis gives us a circle.



$\boxed{\text{Ex 20}}$: $y^2+z^2=18$

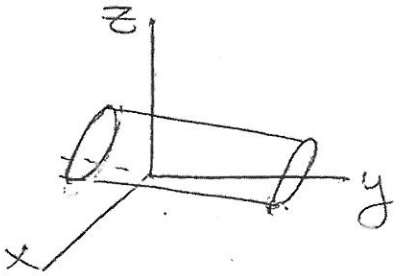
$r=\sqrt{18}$

centered along x-axis, variable not present



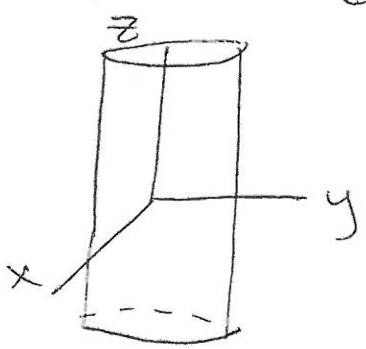
cross-sections along x-axis are circles

$\boxed{\text{Ex 21}}$: $\underbrace{x^2+z^2}_{\text{centered around y-axis}} = 84$ ← $r=\sqrt{84}$



- elliptic cylinders: same as circular but cross-sections are ellipses

$\boxed{\text{Ex 22}}$: $\underbrace{x^2+2y^2}_{\text{centered along z-axis}}=15$



Note: This looks a lot like the others (Ex 19-21), but those cross-sections are circles & this has ellipses

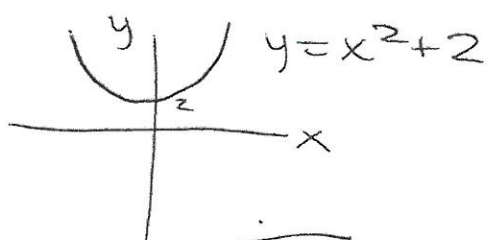
— Parabolic cylinders (the best of both worlds?)

Cross-sections are parabolas

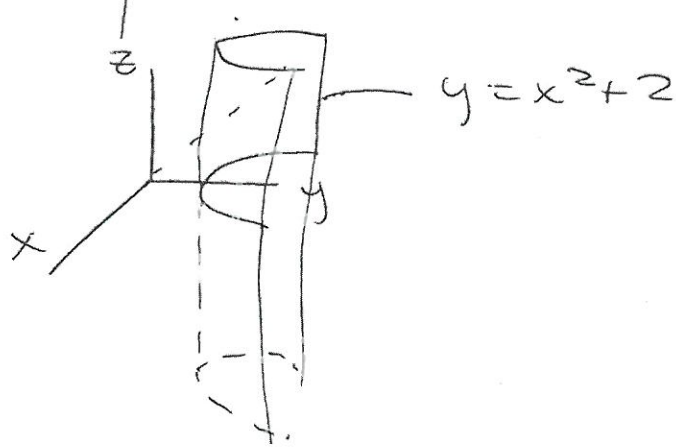
$\boxed{\text{Ex 23}}$: $\underbrace{y = x^2 + 2}$

Centered along variable not present so z-axis in this case

in $\mathbb{R}^2$



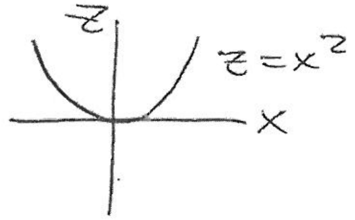
in $\mathbb{R}^3$

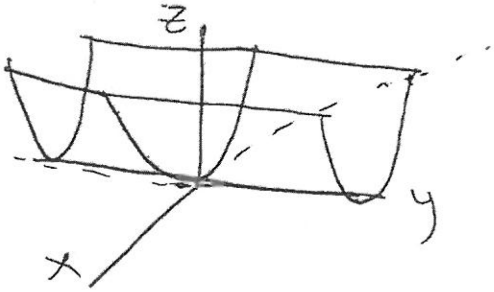


Cross-sections along z-axis are parabolas $y = x^2 + 2$

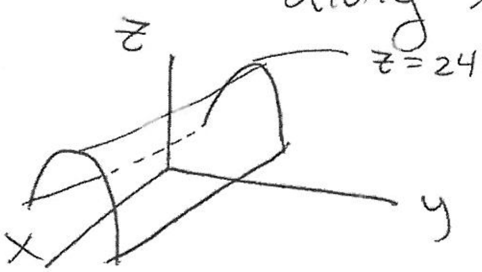
positive opens up

Ex 24: $\underbrace{z = x^2}_{\text{centered along y-axis}}$





Ex 25: $\underbrace{z = 24 - y^2}_{\text{along x-axis}}$ negative opens down



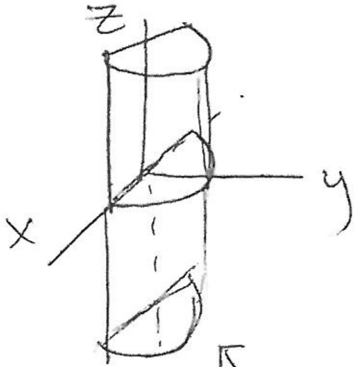
— half a circular cylinder

Ex 26: This is a variation on Ex 19
$y = \sqrt{9 - x^2}$

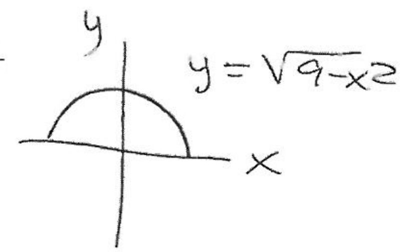
$$\underbrace{y=\sqrt{9-x^2}}$$

$y \geq 0$

still along z-axis
w/ $r=3$



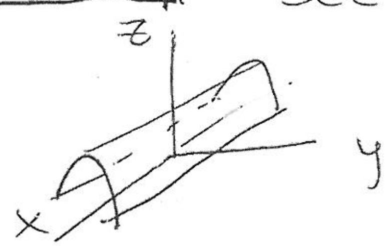
In $\mathbb{R}^2$



looks like parabolic cylinder

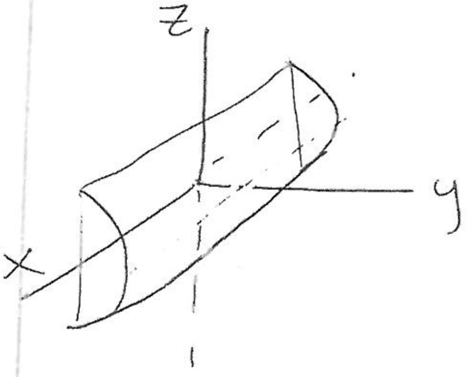
$\boxed{\text{Ex 27}}$: See Ex 20, $\underbrace{z=\sqrt{18-y^2}}_{\text{along } x\text{-axis}}$

$z \geq 0$



$\boxed{\text{Ex 28}}$: See Ex 20 $\underbrace{y=\sqrt{18-z^2}}_{\text{along } x\text{-axis}}$

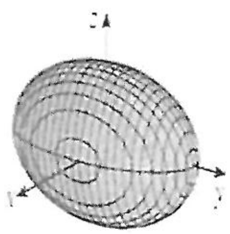
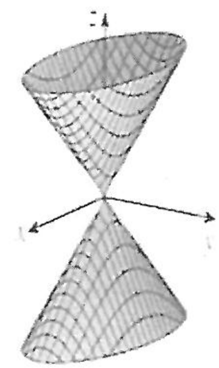
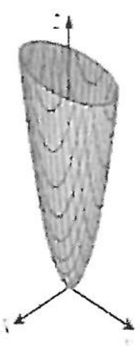
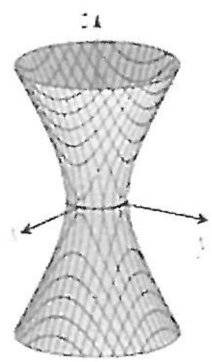
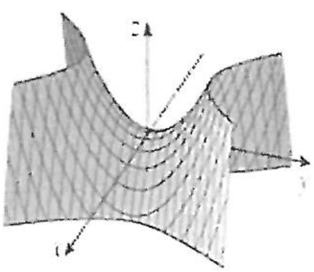
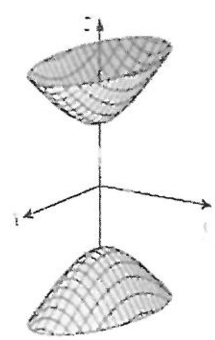
$y \geq 0$



For reference, last 5 interesting but I don't expect you to draw. 1 16

Quadratic Surfaces

Table 2 Graphs of quadric surfaces

Surface	Equation	Surface	Equation
1 Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	2 Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k=0$.
3 Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	4 Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
5 Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	6 Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.