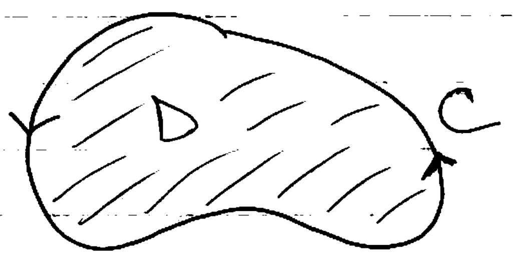
13.4 Green's Theorem

Counterclockwise

Let C be a positively oriented piecewise smooth simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



Examples from Calculus by Larson, Hostetler Edwards

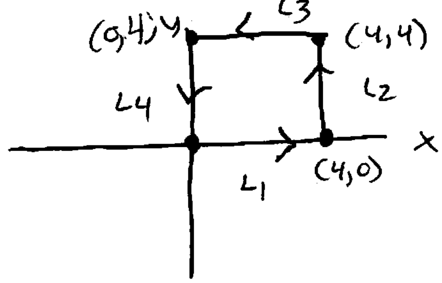
Ex 1 Verify Green's theorem by evaluating both integrals

$$\int_C y^2\,dx + x^2\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

for the indicated path

C: square with vertices (0,0), (4,0), (4,4) (0,4)

P2



Without Green's theorem we need to do 4 separate line integrals

L_1: $y=0,\ x=t \quad 0 \le x \le 4 \rightarrow 0 \le t \le 4$

$$\int_0^4 0^2(1dt) + t^2(0dt) = 0$$

$\underbrace{1dt}_{dx=x'(t)dt}$ $\quad$ $\underbrace{0dt}_{dy=y'(t)dt}$

* Note: we could have done $\vec{r}(t) = (1-t)\vec{r}_0 + \vec{r}_1 t$

$\vec{r}(t) = (1-t)\langle 0,0\rangle + \langle 4,0\rangle t = \langle 4t,0\rangle \quad 0 \le t \le 1$

$\int_0^1 0^2(4dt) + (4t)^2(0dt)$

and gotten the same result

L_2: $x=4 \quad y=t \quad 0 \le y \le 4 \rightarrow 0 \le t \le 4$

$$\int_0^4 t^2(0dt) + 4^2(1dt) = \int_0^4 16dt = 16 \cdot 4 = 64$$

*Alternatively $\vec{r}(t) = (1-t)\langle 4,0\rangle + \langle 4,4\rangle t$

$= \langle 4, 4t\rangle \quad 0 \le t \le 1$

$$\int_0^1 (4t)^2(0dt) + 16(4dt) = \int_0^1 16 \cdot 4dt = 64 \checkmark$$

because these are vertical & horizontal lines it is just as easy not to use $\vec{r}(t)$

L_3: $x=t \quad 4 \le t \le 0 \quad y=4$

$$\int_4^0 4^2(1dt) + t^2(0dt) = -\int_0^4 16dt = -64$$

P3

L_4: $x=0,\ y=t \qquad 4 \le t \le 0$

$$\int_4^0 t^2(0\,dt) + 0(1\,dt) = 0$$

$$\int_C y^2\,dx + x^2\,dy = 0$$

<u>With Green's theorem:</u>

$$\iint_D (\underbrace{2x}_{\frac{\partial Q}{\partial x}} - \underbrace{2y}_{\frac{\partial P}{\partial y}})\,dA$$

the square

$$\int_0^4 \int_0^4 2x - 2y \; dy\,dx = \int_0^4 2xy - y^2 \Big|_0^4 dx$$

$$= \int_0^4 8x - 16\,dx = 4x^2 - 16x \Big|_0^4 = 64 - 64 = 0 \checkmark$$

Ex 2. Use Green's theorem to evaluate the line integral

$$\int_C 2xy\,dx + (x+y)\,dy$$

C: the boundary of the region lying between the graphs $y=0$ and $y=4-x^2$

$$\int_C 2xy\,dx + (x+y)\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \iint_D 1 - 2x\,dA = \int_{-2}^{2} \int_0^{4-x^2} 1 - 2x\,dy\,dx$$

P4

$$\int_{-2}^{2} (1-2x)y \Big|_0^{4-x^2} dx$$

$$\int_{-2}^{2} (1-2x)(4-x^2)\, dx$$

$$= \int_{-2}^{2} (4-8x-x^2+2x^3)\, dx$$

odd functions
$f(-x)=-f(x)$
$\int_{-a}^{a} f(x)\,dx = 0$

$$= \int_{-2}^{2} 4-x^2\,dx = 2\int_0^2 4-x^2\,dx = 2\left[4x-\tfrac{1}{3}x^3\right]_0^2$$

even functions $f(-x)=f(x)$
$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$$= 2\left(8-\tfrac{8}{3}\right) = \boxed{\tfrac{32}{3}}$$

Use Green's theorem to calculate the work done by the force $\vec{F}$ on a particle that is moving counterclockwise around the path C

Ex 3 $\vec{F}(x,y) = xy\hat{\imath} + (x+y)\hat{\jmath}$
$C: x^2+y^2=4$

* We know $W = \int_C \vec{F}\cdot d\vec{r}$ and if
$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ and $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ then
$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \langle P,Q\rangle \cdot \langle x'(t), y'(t)\rangle\, dt = \int_C P\,dx + Q\,dy$

P5

without Green's thm

We need a parametric representation for $x^2+y^2=4$

$x=2\cos t \qquad \vec{r}(t)=\langle 2\cos t, 2\sin t\rangle$

$y=2\sin t \qquad \vec{r}\,'(t)=\langle -2\sin t, 2\cos t\rangle$

$$\int_0^{2\pi} \langle \underbrace{(2\cos t)}_{x}\underbrace{(2\sin t)}_{y}, \underbrace{2\cos t}_{x} + \underbrace{2\sin t}_{y}\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt$$

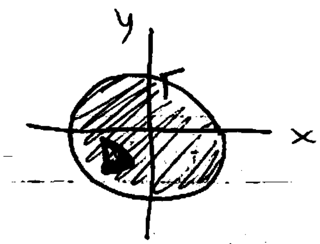
$\langle xy, x+y\rangle$

$$= \int_0^{2\pi} \underbrace{-8\cos t\sin^2 t}_{u=\sin t} + \underbrace{4\cos^2 t}_{\frac{1}{2}(1+\cos 2t)} + \underbrace{4\sin t\cos t}_{u=\sin t}\, dt$$

$$= \int_0^{2\pi} 4\left(\tfrac{1}{2}(1+\cos 2t)\right) dt$$

$$= \int_0^{2\pi} 2 + 2\cos 2t\, dt = 2t + \sin 2t \Big|_0^{2\pi} = \boxed{4\pi}$$

OR with Green's theorem



$P = xy \qquad Q = x+y$

$\dfrac{\partial P}{\partial y} = x \qquad \dfrac{\partial Q}{\partial x} = 1$

$W = \int_C \vec{F}\cdot d\vec{r} = \iint_D 1 - x\, dA$

circular use polar

$$\int_0^{2\pi}\int_0^2 1 - r\cos\theta \; r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 r - r^2\cos\theta\, dr\, d\theta = \int_0^{2\pi} \tfrac{1}{2}r^2 - \tfrac{1}{3}r^3\cos\theta \Big|_0^2 d\theta$$

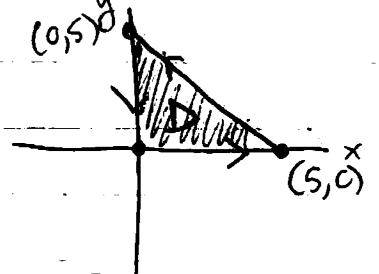
$$= \int_0^{2\pi} 2 - \tfrac{8}{3}\cos\theta\, d\theta = 2\theta - \tfrac{8}{3}\sin\theta\Big|_0^{2\pi} = \boxed{4\pi}$$

✓

P6

Ex 4 $\vec{F}(x,y) = (x^{3/2} - 3y)\hat{\imath} + (6x + 5\sqrt{y})\hat{\jmath}$

C: boundary of the triangle with vertices (0,0), (5,0), and (0,5)



Green's thm: $P = x^{3/2} - 3y \qquad Q = 6x + 5\sqrt{y}$

$$\frac{\partial P}{\partial y} = -3 \qquad \frac{\partial Q}{\partial x} = 6$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= \iint_D 6 - (-3)\, dA$$

$$= \iint_D 9\, dA = 9 \iint_D 1\, dA$$

$$= 9\, A(D) = 9\left(\tfrac{1}{2} bh\right)$$

$$= 9\left(\tfrac{1}{2} 5 \cdot 5\right)$$

$$= \boxed{225/2}$$

* Note: For the previous problems if we were moving clockwise we would have needed to multiply our answers by -1.