Integration By Parts

The goal of integration by parts is to undo the product rule:

$$\frac{d}{dx}[fg] = f'g + fg'$$

$$\int \frac{d}{dx}[fg]\,dx = \int f'g\,dx + \int fg'\,dx$$

$$fg = \int f'g\,dx + \int fg'\,dx$$

OR

$$\int fg'\,dx = fg - \int f'g\,dx$$

But we usually write the formula as

$$\boxed{\int u\,dV = uv - \int v\,du}$$

where $u = f$ so $du = f'dx$
$v = g$ $\quad dv = g'\,dx$

To find u, use LIATE

L – logs
I – inverse trig
A – algebraic (polynomials & things like polynomials ex x^{-2}, $\sqrt[3]{x}$, etc)
T – trig
E – exponentials

By picking u in this order we make dV easy (relatively easy) to integrate.

<u>Technique</u>: $\int u\,dV = uv - \int v\,du$

1. Pick u by LIATE
2. dV is everything left over
3. Plug into the formula

✳ There are lots of times when integration by parts works, but <u>mostly</u> we do it for problems of the following types:

$\int x^n \cos x\,dx$

$\int x^n \sin x\,dx$

$\int x^n \ln x\,dx$ (But <u>not</u> $\int \frac{\ln x}{x}dx$ this is a u-sub w/ $u=\ln x$)

$\int x^n e^x\,dx$

$\int e^x \sin x\,dx$

$\int e^x \cos x\,dx$

these are a hassle but strangely enjoyable. You should have done one of them in Calc 1. Unfortunately they are too hard for a test. :(

Ex 1

$$\int (x+2)\cos x \, dx$$

* we can tell that it is integration by parts since it is of the type $\int x^n \cos x \, dx$

→ Pick u by LIATE

no logs

no inverse trig

yes

differentiate

$u = x+2$

$du = dx$

$v = \sin x$

$dv = \underbrace{\cos x \, dx}$

take antiderivative of dV to find V

everything left after we find u

$$\int u \, dV = uv - \int v \, du$$

$$\int (x+2)\cos x \, dx = (x+2)\sin x - \int \sin x \, dx$$

$$= \boxed{(x+2)\sin x + \cos x + C}$$

Ex 2

$$\int x \sin(3x) \, dx$$

* Of the type $\int x^n \sin x \, dx$

→ Pick u by LIATE

take derivative

$u = x$

$du = dx$

$v = -\frac{1}{3}\cos(3x)$

$dv = \sin(3x)dx$

take the antiderivative

$$\int u \, dV = uv - \int v \, du$$

$$\int x \sin(3x) dx = -\frac{x}{3}\cos(3x) - \int -\frac{1}{3}\cos(3x) \, dx$$

$$= -\frac{x}{3}\cos(3x) + \int \frac{1}{3}\cos 3x \, dx = \boxed{-\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x + C}$$

$\boxed{\text{Ex 3}}$ $\int (x^3+x^2)\ln x\, dx$

→ Pick u by LIATE

so $u = \ln x$ $\qquad v = \left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right)$ take antiderivative

take derivative $\quad du = \frac{1}{x}dx \qquad dV = (x^3+x^2)\,dx$

everything left after we find u

$$\int (x^3+x^2)\ln x\, dx = uv - \int v\, du$$

$$= (\ln x)\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right) - \int \left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right)\frac{1}{x}dx$$

$$= \ln x\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right) - \int \frac{1}{4}x^3 + \frac{1}{3}x^2\, dx$$

$$= \ln x\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right) - \left(\frac{1}{16}x^4 + \frac{1}{9}x^3\right) + C$$

$$= \boxed{\ln x\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right) - \frac{1}{16}x^4 - \frac{1}{9}x^3 + C}$$

* $\int x^n \ln x\, dx$ is one of the easiest integration by parts problems, no matter how weird we make x^n it is still easy to do.

$\boxed{\text{Ex 4}}$ $\int_1^4 \sqrt{x}\ln x\,dx$

Definite integrals (integrals w/ end points) do make integration by parts a little harder. Basically all we need to do is evaluate after we integrate.

→ Pick u by LIATE

$u = \ln x \qquad V = \frac{2}{3}x^{3/2}$

$du = \frac{1}{x}dx \qquad dV = \sqrt{x}\,dx = x^{1/2}dx$

$$\int_1^4 \sqrt{x}\ln x\,dx = \underbrace{\frac{2}{3}x^{3/2}}_{V}\underbrace{\ln x}_{u}\Big|_1^4 - \int_1^4 \underbrace{\frac{2}{3}x^{3/2}}_{V}\underbrace{\frac{1}{x}dx}_{du}$$

$$\frac{2}{3}(4)^{3/2}\ln 4 - \cancel{\frac{2}{3}1^{3/2}\ln 1} - \int_1^4 \frac{2}{3}x^{1/2}dx$$

$$\frac{2}{3}(8)\ln 4 - \frac{4}{9}x^{3/2}\Big|_1^4$$

$$\frac{16}{3}\ln 4 - \left(\frac{4}{9}\cdot 4^{3/2} - \frac{4}{9}\cdot 1^{3/2}\right)$$

$$\boxed{\frac{16}{3}\ln 4 - \frac{4}{9}(8) + \frac{4}{9}}$$

$\boxed{\text{Ex 5}}$

$$\int_0^1 xe^{-2x}dx$$

→ Choose u by LIATE

$u=x \qquad V=-\frac{1}{2}e^{-2x}$

$du=dx \qquad dV=e^{-2x}dx$

$$\int_0^1 xe^{-2x}dx = -\frac{x}{2}e^{-2x}\Big|_0^1 - \int_0^1 -\frac{1}{2}e^{-2x}dx$$

$$uv\Big|_0^1 - \int_0^1 v\,du$$

$$= -\frac{1}{2}e^{-2} - \cancel{\left(-\frac{0}{2}e^{-2\cdot 0}\right)} + \int_0^1 \frac{1}{2}e^{-2x}dv$$

$$= -\frac{1}{2e^{2}} - \frac{1}{4}e^{-2x}\Big|_0^1$$

$$-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4}e^{-2\cdot 0}$$

$$-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4}$$

$$\boxed{-\frac{3}{4}e^{-2} + \frac{1}{4}}$$

$\boxed{\text{Ex 6}}$ $\int (x^2+2)e^x\,dx$

* This would be an ideal candidate for Tabular Integration by parts

Short cut that works if u = polynomial

→ Choose u by LIATE

$u = x^2+2$ $\quad v = e^x$

$du = 2x\,dx$ $\quad dv = e^x dx$

$$\int (x^2+2)e^x dx = uv - \int v\,du$$
$$= (x^2+2)e^x - \int e^x(2x)\,dx$$

to integrate this we do integration by parts again!

$$\int e^x(2x)\,dx$$

$u = 2x$ $\quad v = e^x$

$du = 2dx$ $\quad dv = e^x dx$

(you could also do $u = x$ & $dv = 2e^x$ & get the same result)

$$\int e^x(2x)\,dx = uv - \int v\,du$$
$$= 2xe^x - \int e^x 2\,dx$$
$$= 2xe^x - 2e^x + C$$

Plug back in!

$$\int (x^2+2)e^x dx = (x^2+2)e^x - \int e^x 2x\,dx$$
$$= (x^2+2)e^x - [2xe^x - 2e^x + C]$$
$$= \boxed{(x^2+2)e^x - 2xe^x + 2e^x + C}$$

C is just a constant so $-C$ would still just be a constant