Limits

For a surface $z = f(x,y)$ we want to be able to find $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$.

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ is not the same for all approaches or paths within the domain of f, the limit does not exist.

To show a limit doesn't exist, we're looking for a contradiction. Basically we approach the point along 2 different paths & get 2 different answers.

Technique: If $\lim_{(x,y)\to(0,0)} f(x,y)$

1. Try $x=0, y\to 0$
2. Try $y=0, x\to 0$
3. Try combining the denominator. Can you make $x=f(y)$ or $y=g(x)$ so that the terms of the denominator can be added together?

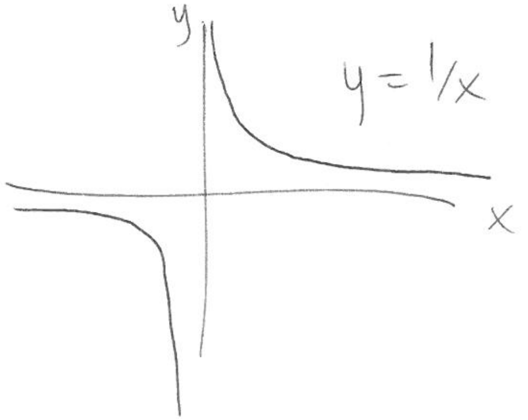
$\boxed{\text{Ex 1}}$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y^2}{x^2+y^2}$

☆ Note: we'd get $\frac{0}{0}$ if we were to plug in

Let $x=0$ $\displaystyle\lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1$

Let $y=0$ $\displaystyle\lim_{(x,0)\to(0,0)} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} \to$ does not exist

DNE



$\displaystyle\lim_{x\to 0^+} \frac{1}{x} \to \infty$

$\displaystyle\lim_{x\to 0^-} \frac{1}{x} \to -\infty$

Since our limits don't agree, the $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y^2}{x^2+y^2}$ DNE

Ex 2

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$$

$$x=0: \lim_{(0,y)\to(0,0)} \frac{0^2y^2}{0^4+y^4} = \lim_{y\to 0} \frac{0}{y^4} = 0$$

If we tried $y=0$, we'd get the same answer of 0. That is fine, but it isn't helpful. Instead, we'll try to combine the denominator. We want both the terms of the denominator to be x^4 or y^4 so in this case we'll try

$$y=x: \lim_{(x,x)\to(0,0)} \frac{x^2x^2}{x^4+x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

$$\underbrace{0 \neq \tfrac{1}{2} \text{ limit DNE}}_{\text{make sure you write this}}$$

$\boxed{\text{Ex 3}}$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y^3}{(x^2+y^6)^2}$

$x=0$: $\lim\limits_{(0,y)\to(0,0)} \dfrac{0}{(y^6)^2} = 0$

Likewise if $y=0$ so you only need to show one of them

To combine the denominator here we either want to have both terms be x^2 or both be y^6.

If $x^2=y^6$ then $x=\sqrt{y^6}=y^3$
(If we did $x=-\sqrt{y^6}$, it should also work but would make things unnecessarily complicated)

$x=y^3$

$\lim\limits_{(y^3,y)\to(0,0)} \dfrac{(y^3)^3y^3}{(y^6+y^6)^2} = \lim\limits_{y\to0} \dfrac{y^9y^3}{(2y^6)^2} = \lim\limits_{y\to0} \dfrac{y^{12}}{4y^{12}} = \dfrac{1}{4}$

$\boxed{0 \neq \frac{1}{4}\text{, limit DNE}}$

$y=\sqrt[3]{x}$ would have been another choice to combine the denominator

To show a limit exists let's focus on 3 main techniques

$\boxed{\text{Ex 4}}$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+4}-2}$

If we plug in (0,0) we get $\frac{0}{0}$.

if you see $\_\_\_\_ \pm \sqrt{}$

or $\sqrt{} \pm \_\_\_\_$, multiply

by the conjugate.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+4}-2} \cdot \frac{\sqrt{x^2+y^2+4}+2}{\sqrt{x^2+y^2+4}+2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{(x^2+y^2)}\left(\sqrt{x^2+y^2+4}+2\right)}{\underbrace{(x^2+y^2+4)-4}_{\cancel{x^2+y^2}}} = \sqrt{4}+2 = \boxed{4}$$

Basically we used algebra to remove the discontinuity & get the answer

$\boxed{\text{Ex 5}}$ $\displaystyle\lim_{(x,y)\to(a,a)} \frac{x^4-y^4}{x^2-y^2}$

Where a is a constant

Note: we can factor & simplify this problem

$$\lim_{(x,y)\to(a,a)} \frac{x^4-y^4}{x^2-y^2} = \lim_{(x,y)\to(a,a)} \frac{(x^2+y^2)\cancel{(x^2-y^2)}}{\cancel{x^2-y^2}}$$

$$= \boxed{a^2+a^2} = 2a^2$$

The 3rd technique we're going to look at to determine a limit is the Squeeze theorem.

If $g(x,y) \leq f(x,y) \leq h(x,y)$

& $\displaystyle\lim_{(x,y)\to(x_0,y_0)} g(x,y) = \lim_{(x,y)\to(x_0,y_0)} h(x,y) = L$

then $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ by the Squeeze theorem

$\boxed{\text{Ex 6}}$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2}$

not necessary

If we tried $x=0$: $\lim\limits_{(0,y)\to(0,0)} \dfrac{0}{y^2}=0$

likewise if $y=0$

If we tried to combine the denominator we would do $y=x$

$$\lim_{(x,x)\to(0,0)} \frac{x^2x^2}{x^2+x^2} = \lim_{x\to 0} \frac{x^4}{2x^2} = \lim_{x\to 0}\frac{x^2}{2}=0$$

☆ We haven't proved that the limit exists yet; however, this is enough for us to think it might exist. If it does, based on the limits we found, then it is 0. ☆

To use the Squeeze theorem we need to find functions to bound ours by.

$$0 \le \frac{x^2y^2}{x^2+y^2} \le \frac{\cancel{x^2}y^2}{\underbrace{\cancel{x^2}}_{}} = y^2 \to 0 \text{ as } (x,y)\to(0,0)$$

we know $\dfrac{x^2y^2}{x^2+y^2}$ can't be negative

smaller denominator means our nonnegative fraction is bigger

so by the squeeze thm $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2}=0$

$\boxed{\text{Ex 7}}$

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$$

$$\underbrace{\frac{-3x^2|y|}{x^2+y^2}} \le \frac{3x^2y}{x^2+y^2} \le \frac{3x^2|y|}{x^2+y^2} \le \underbrace{\frac{3\cancel{x^2}|y|}{\cancel{x^2}}} = 3|y|$$

Unlike the previous ex $\frac{3x^2y}{x^2+y^2}$ can be negative

so $-\frac{3x^2|y|}{x^2+y^2} \le 0$

$\frac{3x^2|y|}{x^2+y^2} \ge 0$

smaller denominator $\to$ whole thing is bigger

so

$$-3|y| = \underbrace{\frac{-3x^2|y|}{x^2}} \le \frac{-3x^2|y|}{x^2+y^2} \le \frac{3x^2y}{x^2+y^2} \le 3|y| \to 0 \text{ as } (x,y)\to(0,0)$$

$-3|y| \to 0$ as $(x,y)\to(0,0)$

this is more negative than

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$ by the

Squeeze thm.