Line Integrals

Formulas: For all of these assume we have a smooth curve C given by $\vec{r}(t)$, $a \le t \le b$

Line Integral of f along C

(Also called the line integral w.r.t. arc length)

$$\boxed{\begin{array}{c} \int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt \\ \text{or} \quad \int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt \end{array}}$$

★ If $f = \rho$ is the linear density of a wire C, then $\int_C \rho(x,y)\,ds$ or $\int_C \rho(x,y,z)\,ds$ will give us the mass of the wire ★

Line Integral of $\vec{F}$ along C

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt}$$

★ This formula also gives us the total work moving a particle along C ★

Note: If $\vec{F} = \langle P, Q \rangle$ and $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P x'(t)\,dt + Q y'(t)\,dt$$

$$= \int_C P\,dx + Q\,dy$$

<u>Line Integrals w.r.t x, y, & z</u>

$$\boxed{\begin{aligned} \int_C f\,dx &= \int_a^b f(\vec{r}(t))\, x'(t)\,dt \\ \int_C f\,dy &= \int_a^b f(\vec{r}(t))\, y'(t)\,dt \\ \int_C f\,dz &= \int_a^b f(\vec{r}(t))\, z'(t)\,dt \end{aligned}}$$

★ These frequently occur together since they originate from $\int_C \vec{F} \cdot d\vec{r}$. You might see $\int_C P\,dx + Q\,dy$ or

$$\int_C P\,dx + Q\,dy + R\,dz$$ ★

Now that we have the formulas, the only thing we need is a parametric representation of C.

Possibilities:

1) They give you the representation (this is the best case scenario)

2) Line segment

$$\vec{r}(t) = (1-t)\vec{r_0} + \vec{r_1}\,t \qquad 0 \le t \le 1$$

start point ($\vec{r_0}$) end point ($\vec{r_1}$)

3) Circle $x^2+y^2=a^2$

$x = a\cos t$
$y = a\sin t$ or $\vec{r}(t) = \langle a\cos t, a\sin t \rangle$

To find the bounds treat t like theta from polar coordinates

4) $y=f(x)$ $(a, f(a))$ to $(b, f(b))$

$x = t$
$y = f(t)$ or $\vec{r}(t) = \langle t, f(t) \rangle$
$a \le t \le b$ $\qquad a \le t \le b$

5) $x = g(y)$ from $(g(c), c)$ to $(g(d), d)$

$x = g(t)$
$y = t$ or $\vec{r}(t) = \langle g(t), t \rangle$ $\quad c \le t \le d$
$c \le t \le d$

There are more possible representations but for the most part, I'll restrict myself to the 5 mentioned.

The following examples mostly come from <u>Calculus 6e</u> by Edwards & Penney

$\boxed{\text{Ex 1}}$ Evaluate the line integral $\int_C f(x,y)\,ds$ along the indicated curve

$f(x,y) = x+y$, $x = e^t+1$, $y = e^t-1$ $\quad 0 \le t \le \ln 2$

* They have given us our representation
$\vec{r}(t) = \langle e^t+1, e^t-1 \rangle \rightarrow \vec{r}\,'(t) = \langle e^t, e^t \rangle$

$$\int_0^{\ln 2} \Big[\underset{\substack{\uparrow \\ x}}{(e^t+1)} + \underset{\substack{\uparrow \\ y}}{(e^t-1)}\Big] \underbrace{\sqrt{(e^t)^2+(e^t)^2}}_{ds}\,dt$$

$f = x+y$

* Simplifying $\int_0^{\ln 2} 2e^t\sqrt{e^{2t}+e^{2t}}\,dt$

$= \int_0^{\ln 2} 2e^t\sqrt{2e^{2t}}\,dt = \int_0^{\ln 2} 2e^t\sqrt{2}\sqrt{e^{2t}}\,dt$

$= \int_0^{\ln 2} 2e^t\sqrt{2}\,e^t\,dt = \int_0^{\ln 2} 2\sqrt{2}\,e^{2t}\,dt$

$$= \sqrt{2}e^{2t}\Big|_0^{\ln 2} = \sqrt{2}\,e^{2\ln 2} - \sqrt{2}e^{0}$$

$$= \sqrt{2}\,e^{\ln 2^2} - \sqrt{2}$$

$$= \sqrt{2}\,4 - \sqrt{2} = \boxed{3\sqrt{2}}$$

fancy

Note: If $\rho = x+y$ we would have found $3\sqrt{2}$ to be the mass of the wire

Ex 2 Evaluate $\int_C y^2dx + x\,dy$ where C is the part of the graph $x = y^3$ from $(-1,-1)$ to $(8,2)$

☆ This is the 5th one on our representation list.

$x = t^3$

$y = t$

$-1 \le t \le 2$ (Note: $y=t$, use the points given to find what y is between)

$$\int_C y^2dx + x\,dy = \int_{-1}^{2} \underbrace{(t)^2}_{y}\underbrace{(3t^2)}_{x'(t)} + \underbrace{(t^3)}_{x}\underbrace{(1)}_{y'(t)}\,dt$$

$$= \int_{-1}^{2} t^2\,3t^2 + t^3\,dt$$

$$= \int_{-1}^{2} 3t^4 + t^3\,dt = \frac{3}{5}t^5 + \frac{1}{4}t^4\Big|_{-1}^{2} = \boxed{\frac{3}{5}2^5 + \frac{1}{4}\cdot 2^4 - \left(\frac{3}{5}(-1) + \frac{1}{4}\right)}$$

$\boxed{\text{Ex 3}}$ Find the mass of the wire with density $\rho(x,y,z) = xyz$ in the shape of the line segment from $(1,-1,2)$ to $(3,2,5)$

* we need $m = \int_C \rho(x,y,z)\,ds$ and the representation for a line segment

$$\vec{r}(t) = (1-t)\vec{r}_0 + \vec{r}_1 t \quad 0 \le t \le 1$$

$$\vec{r}(t) = (1-t)\langle 1,-1,2\rangle + \langle 3,2,5\rangle t$$

* After you plug into our line segment formula, Simplify so you know the values of x, y, & z

$$\vec{r}(t) = \langle 1-t, -1+t, 2-2t\rangle + \langle 3t, 2t, 5t\rangle$$

$$= \langle \underset{x}{\underset{\shortparallel}{1+2t}}, \underset{y}{\underset{\shortparallel}{-1+3t}}, \underset{z}{\underset{\shortparallel}{2+3t}}\rangle$$

$$m = \int_C \rho(x,y,z)\,\underset{|\vec{r}\,'(t)|dt}{ds} = \int_0^1 \underset{x}{(1+2t)}\underset{y}{(-1+3t)}\underset{z}{(2+3t)}\sqrt{(2)^2+(3)^2+(3)^2}\ \underset{dt}{}$$

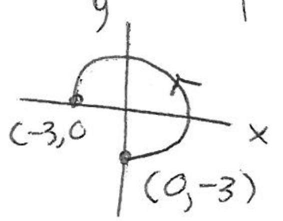
$$= \int_0^1 (1+2t)(-1+3t)(2+3t)\sqrt{22}\,dt$$

To finish distribute & integrate. I wouldn't ask something quite this tedious.

$\boxed{\text{Ex 4}}$ Kurtz ☺ a) Find the work done by $\vec{F}(x,y) = \langle xy, -x^2 \rangle$ moving a particle from $(0,-3)$ to $(-3,0)$ along the circle $x^2+y^2=9$

☆ We need the formula $W = \int_C \vec{F} \cdot d\vec{r}$

& a representation for the circle



$$x = 3\cos t$$
$$y = 3\sin t$$
$$-\frac{\pi}{2} \le t \le \pi$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_{-\frac{\pi}{2}}^{\pi} \langle \underbrace{3\cos t}_{x}\, \underbrace{3\sin t}_{y}, \underbrace{-9\cos^2 t}_{-x^2} \rangle \cdot \langle \underbrace{-3\sin t}_{x'(t)}, \underbrace{3\cos t}_{y'(t)} \rangle\, dt$$

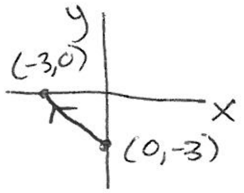
$$= \int_{-\frac{\pi}{2}}^{\pi} -27\cos t \sin^2 t - 27\cos^3 t \, dt$$

$$= \int_{-\frac{\pi}{2}}^{\pi} -27\cos t\,(\sin^2 t + \cos^2 t)\, dt$$

$$= \int_{-\frac{\pi}{2}}^{\pi} -27\cos t\, dt = -27\sin t \Big|_{\frac{\pi}{2}}^{\pi}$$

$$\cancel{-27\sin\pi} - \left(-27\sin\frac{\pi}{2}\right) = \boxed{27}$$

b) Find the work done by $\vec{F}(x,y) = \langle xy, -x^2 \rangle$ moving a particle along the line segment from $(0,-3)$ to $(-3,0)$



Line segment representation

$$\vec{r}(t) = (1-t)\langle 0,-3\rangle + \langle -3,0\rangle t$$
$$= \langle 0, -3+3t\rangle + \langle -3t, 0\rangle$$
$$= \langle \underset{x}{\underset{\|}{-3t}}, \underset{y}{\underset{\|}{-3+3t}} \rangle$$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle \underset{x}{(-3t)}\underset{y}{(-3+3t)}, -\underset{-x^2}{(-3t)^2} \rangle \cdot \langle \underset{x'}{-3}, \underset{y'}{3} \rangle \, dt$$

$$= \int_0^1 \langle 9t-9t^2, -9t^2 \rangle \cdot \langle -3, 3\rangle \, dt$$

$$= \int_0^1 -27t + \cancel{27t^2} - \cancel{27t^2} \, dt$$

$$= -\frac{27}{2}t^2 \Big|_0^1 = \boxed{-\frac{27}{2}}$$

c) Is this line integral path independent?

No, we got different answers for different paths w/ same initial & terminal points.
(To show path independence show $\text{curl}\,\vec{F} = \vec{0}$)

$\boxed{\text{Ex 5}}$ Evaluate $\int_C \sqrt{z}\,dx + \sqrt{x}\,dy + y^2\,dz$
where C is the curve $x=t,\ y=t^{3/2},\ z=t^2$
$0 \le t \le 4$

$$\int_0^4 \sqrt{t^2}\,\underbrace{1}_{x'} + \sqrt{t}\,\underbrace{\left(\tfrac{3}{2}t^{1/2}\right)}_{y'} + \left(t^{3/2}\right)^2 \underbrace{2t}_{z'}\ dt$$

$$= \int_0^4 t + \frac{3}{2}t + t^3 2t\,dt$$

$$= \int_0^4 \frac{5}{2}t + 2t^4\,dt = \frac{5}{4}t^2 + \frac{2}{5}t^5 \Big|_0^4$$

$$= \boxed{20 + \frac{2}{5}\cdot 4^5}$$

$\boxed{\text{Ex 6}}$ Find the work done by the force field $\vec{F} = \langle z, -x, y\rangle$ in moving a particle from $(1,1,1)$ to $(2,4,8)$ along the curve $y=x^2,\ z=x^3$

* Note: both y & z are functions of x
So our representation is

$x=t$
$y=t^2$ or $\vec{r}(t) = \langle t, t^2, t^3\rangle \quad 1 \le t \le 2$
$z=t^3 \quad 1 \le t \le 2$

$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_1^2 \langle t^3, -t, t^2 \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

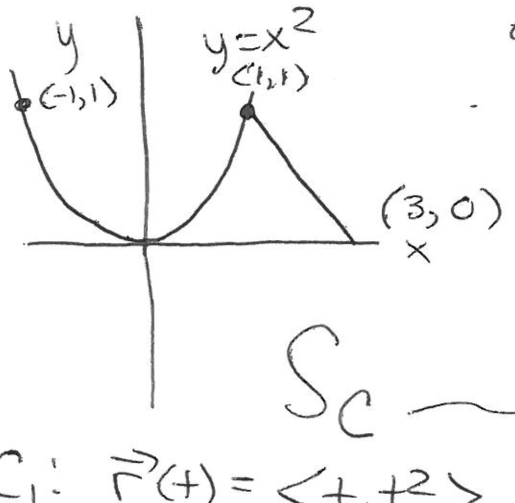
$z \quad -x \quad y$

$$= \int_1^2 t^3 - 2t^2 + 3t^4 \, dt$$

$$= \frac{1}{4}t^4 - \frac{2}{3}t^3 + \frac{3}{5}t^5 \Big|_1^2$$

$$= \boxed{\frac{1}{4}\cdot 2^4 - \frac{2}{3}\cdot 2^3 + \frac{3}{5}\cdot 2^5 - \left(\frac{1}{4} - \frac{2}{3} + \frac{3}{5}\right)}$$

- One last thing for piecewise smooth curves, evaluate your line integrals in pieces



Ex

If C is the parabola $y=x^2$ from $(-1,1)$ to $(1,1)$ and the line segment from $(1,1)$ to $(3,0)$ then

$$\int_C \sim = \int_{C_1} \sim + \int_{C_2} \sim$$

C_1: $\vec{r}(t) = \langle t, t^2 \rangle \quad -1 \le t \le 1$

C_2: $\vec{r}(t) = (1-t)\langle 1,1 \rangle + \langle 3,0 \rangle t = \langle 1+2t, 1-t \rangle \quad 0 \le t \le 1$