Finding Inverses using row operations

There are lots of uses for inverses of matrices.

To find A^{-1}, augment A with the identity matrix, perform row operations until A becomes the identity matrix.

$$[A \mid I] \longrightarrow [I \mid A^{-1}]$$

If this can't be done, A is not invertible (doesn't have an inverse).
(Recall if $\det A = 0$, A doesn't have an inverse)

The 3 row operations we are allowed are
1. swapping rows
2. multiplying a row by a nonzero constant
3. adding a constant multiple of one row to another.

I will find the inverses using Gauss-Jordan Elimination (this is not the only method that works). Although this isn't the fastest method, it is good if you are a beginner.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & & & & \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Step 1) make $a_{11} = 1$ using row operations

Step 2) get the entries beneath a_{11} equal to zero using row ops

Step 3) Get $a_{22} = 1$ using row ops

Step 4) Get the entry above a_{22} & the entries beneath a_{22} to be zero

Step 5) get $a_{33} = 1$ using row ops

Step 6) Get the entries above & beneath it to be zero

Continue until you have an identity matrix (If $\det A = 0$, this is impossible)

Find A^{-1} using row operations

Ex 1: $A = \begin{bmatrix} 4 & 1 \\ 5 & 9 \end{bmatrix}$

Augment: $\left[\begin{array}{cc|cc} \boxed{4} & 1 & 1 & 0 \\ 5 & 9 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{4}r_1} \left[\begin{array}{cc|cc} 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ \boxed{5} & 9 & 0 & 1 \end{array}\right]$

make this a 1

make this a 0

$r_2 - 5r_1 \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \boxed{\frac{31}{4}} & -\frac{5}{4} & 1 \end{array}\right] \xrightarrow{\frac{4}{31}r_2} \left[\begin{array}{cc|cc} 1 & \boxed{\frac{1}{4}} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{5}{31} & \frac{4}{31} \end{array}\right]$

make this a 1

$\frac{36}{4} - \frac{5}{4}$

make this a 0

$r_1 - \frac{1}{4}r_2$

$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{9}{31} & -\frac{1}{31} \\ 0 & 1 & -\frac{5}{31} & \frac{4}{31} \end{array}\right]$

$\frac{1}{4} - \frac{1}{4}\left(\frac{-5}{31}\right)$

$$\boxed{A^{-1} = \begin{bmatrix} \frac{9}{31} & -\frac{1}{31} \\ -\frac{5}{31} & \frac{4}{31} \end{bmatrix}}$$

Ex 1 → Bonus question

If they said solve $\begin{bmatrix} 4 & 1 \\ 5 & 9 \end{bmatrix}\vec{x} = \begin{bmatrix} 31 \\ 62 \end{bmatrix}$ for $\vec{x}$ we would do $A^{-1}A\vec{x} = A^{-1}\begin{bmatrix} 31 \\ 62 \end{bmatrix}$

Bonus meaning additional not points :;

or $\vec{x} = A^{-1}\begin{bmatrix} 31 \\ 62 \end{bmatrix}$

$$= \begin{bmatrix} \frac{9}{31} & \frac{-1}{31} \\ \frac{-5}{31} & \frac{4}{31} \end{bmatrix} \begin{bmatrix} 31 \\ 62 \end{bmatrix}$$

$$= \begin{bmatrix} (9-2) \\ (-5+8) \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\boxed{x_1 = 7, \ x_2 = 3}$$

Ex 2: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

need this to be a 1 & it is woohoo!

need this to be a zero

$$\xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_3 - r_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{-2} & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right]$$

need this to be a zero

need this to be a 1, we can't add a multiply of r_1 without messing up a_{21}

need this to be zero

$-\frac{1}{2}r_2$ $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right]$

$r_1 - r_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \boxed{3} & -1 & 0 & 1 \end{array}\right]$

already 0!

need this to be 1

$\frac{1}{3}r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & & & \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right]$

need this to be 0

$r_2 + \frac{1}{2}r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right]$

$\frac{1}{2}$ $-\frac{1}{6}$

need this to be zero

$r_1 - \frac{3}{2}r_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right]$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}}$$