Directional Derivatives

- If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b \rangle$ and

$$\boxed{D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b}$$
$$\boxed{= \nabla f(x,y) \cdot \vec{u}}$$

or if f is a function of x, y, z

$$\boxed{\begin{aligned} D_u f(x,y,z) &= f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c \\ &= \nabla f(x,y,z) \cdot \vec{u} \end{aligned}}$$

Ex p799 Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$

11. $f(x,y) = 1 + 2x\sqrt{y}$ $\quad (3,4) \quad \vec{v} = \langle 4, -3 \rangle$

First we calculate the gradient and plug in our point: $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle 2\sqrt{y}, 2x\frac{1}{2}y^{-1/2} \rangle$

$$= \langle 2\sqrt{y}, xy^{-1/2} \rangle$$

$$\nabla f(3,4) = \langle 2\sqrt{4}, 3(4)^{-1/2} \rangle = \langle 4, \tfrac{3}{2} \rangle$$

Next we need to find a unit vector in the same direction as $\vec{v}$. So $\vec{u} = \frac{\vec{v}}{|\vec{v}|}$.

$$\vec{u} = \frac{\langle 4,-3\rangle}{\sqrt{4^2+(-3)^2}} = \frac{\langle 4,-3\rangle}{5} = \left\langle \frac{4}{5}, \frac{-3}{5}\right\rangle$$

$$D_uf(3,4) = \nabla f(3,4) \cdot \vec{u} = \left\langle 4, \frac{3}{2}\right\rangle \cdot \left\langle \frac{4}{5}, \frac{-3}{5}\right\rangle$$
$$= \frac{16}{5} - \frac{9}{10} = \frac{32-9}{10} = \boxed{\frac{23}{10}}$$

15. $g(x,y,z) = (x+2y+3z)^{3/2}$ $(1,1,2)$ $\vec{v} = 2\hat{j} - \hat{k}$

→ First find gradient: $\nabla g = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle$

$$\nabla g = \left\langle \frac{3}{2}(x+2y+3z)^{1/2} \cdot 1, \frac{3}{2}(x+2y+3z)^{1/2} 2, \frac{3}{2}(x+2y+3z)^{1/2} 3\right\rangle$$

$$\nabla g(1,1,2) = \left\langle \frac{3}{2}(1+2+6)^{1/2}, 3(1+2+6)^{1/2}, \frac{9}{2}(1+2+6)^{1/2}\right\rangle$$
$$= \left\langle \frac{9}{2}, 9, \frac{27}{2}\right\rangle$$

→ Make $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 0,2,-1\rangle}{\sqrt{0^2+2^2+(-1)^2}} = \frac{\langle 0,2,-1\rangle}{\sqrt{5}} = \left\langle 0, \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right\rangle$

→ $D_ug(1,1,2) = \nabla g(1,1,2) \cdot \vec{u} = \left\langle \frac{9}{2}, 9, \frac{27}{2}\right\rangle \cdot \left\langle 0, \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right\rangle$

$$= \frac{9}{2}(0) + 9\left(\frac{2}{\sqrt{5}}\right) + \frac{27}{2}\left(\frac{-1}{\sqrt{5}}\right)$$
$$= \frac{18}{\sqrt{5}} - \frac{13.5}{\sqrt{5}} = \boxed{\frac{4.5}{\sqrt{5}}}$$

Maximizing the Directional Derivative

Thm: Suppose f is a differentiable function of 2 or 3 variables. The maximum value of the directional derivative $D_u f(x,y)$ (or $D_u f(x,y,z)$) is $|\nabla f(x,y)|$ (or $|\nabla f(x,y,z)|$) and it occurs when $\vec{u}$ has the same direction as ∇f.

What this means:
- The direction of the maximum rate of change is ∇f evaluated at our point
- The maximum rate of change is $|\nabla f|$ evaluated at our point.

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47. Find the maximum rate of change of $f(x,y) = x^2 y + \sqrt{y}$ at the point $(2,1)$. In which direction does it occur?

→ First we find the gradient

$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle 2xy, x^2 + \frac{1}{2}y^{-1/2} \rangle$

→ $\nabla f(2,1) = \langle 2(2)(1), 2^2 + \frac{1}{2}(1)^{-1/2} \rangle$

$= \langle 4, 4.5 \rangle$

$\boxed{\text{The direction of the maximum rate of change is } \nabla f(2,1) = \langle 4, 4.5 \rangle}$

★ If they had asked for this as a unit vector (or as a vector of magnitude 1), our answer would have been $\frac{\nabla f(2,1)}{|\nabla f(2,1)|} = \frac{\langle 4, 4.5 \rangle}{\sqrt{4^2 + (\frac{9}{2})^2}} = \frac{\langle 4, 4.5 \rangle}{\sqrt{16 + \frac{81}{4}}}$

$\frac{9}{2} = 4.5$

$= \frac{\langle 4, 4.5 \rangle}{\sqrt{\frac{64}{4} + \frac{81}{4}}} = \frac{\langle 4, 4.5 \rangle}{\sqrt{\frac{145}{4}}} = \frac{2\langle 4, 4.5 \rangle}{\sqrt{145}}$ ★

$\boxed{\text{The maximum rate of change of } f \text{ at } (2,1) \text{ is } |\nabla f(2,1)| = \sqrt{\frac{145}{4}} = \frac{\sqrt{145}}{2}}$

Ex From Calculus 6e by Edwards & Penney

p916 Find the maximum directional derivative of f at P and the direction in which it occurs

21. $f(x,y) = 2x^2 + 3xy + 4y^2 \quad P(1,1)$

→ Find the gradient $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

$= \langle 4x + 3y, 3x + 8y \rangle$

→ Find the gradient at (1,1)

$\nabla f(1,1) = \langle 4+3, 3+8 \rangle$

$= \langle 7, 11 \rangle$

The direction of the maximum directional derivative is $\nabla f(1,1) = \langle 7, 11 \rangle$

The maximum directional derivative is $|\nabla f(1,1)| = |\langle 7,11 \rangle| = \sqrt{49+121} = \sqrt{170}$

Alternatively we could have $\vec{u} = \dfrac{\nabla f(1,1)}{|\nabla f(1,1)|} = \dfrac{\langle 7,11 \rangle}{\sqrt{170}}$

$D_u f(1,1) = \nabla f(1,1) \cdot \vec{u} = \langle 7, 11 \rangle \cdot \dfrac{\langle 7,11 \rangle}{\sqrt{170}}$

$= \dfrac{49}{\sqrt{170}} + \dfrac{121}{\sqrt{170}} = \dfrac{170}{\sqrt{170}} \cdot 1 = \dfrac{170}{\sqrt{170}} \dfrac{\sqrt{170}}{\sqrt{170}} = \sqrt{170}$

Ex Find the maximum rate of change of the function at the given point. In which direction does it occur?

$F(x,y,z) = e^{x-y-z}$ $P(5,2,3)$

$\rightarrow$ Find $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$= \langle e^{x-y-z} 1, e^{x-y-z}(-1), e^{x-y-z}(-1) \rangle$

$= \langle e^{x-y-z}, -e^{x-y-z}, -e^{x-y-z} \rangle$

$\rightarrow$ $\nabla f(5,2,3) = \langle e^{5-2-3}, -e^{5-2-3}, -e^{5-2-3} \rangle$

$= \langle e^0, -e^0, -e^0 \rangle$

$= \langle 1, -1, -1 \rangle$

The direction of the maximum rate of change $\nabla f(5,2,3) = \langle 1,-1,-1 \rangle$

$\rightarrow$ The maximum rate of change is $|\nabla f(5,2,3)| = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}$