Maximum and Minimum Values

2nd Derivative Test: Suppose the 2nd partial derivatives of f are continuous on a disk with center (a,b) and suppose $f_x(a,b)=0$ and $f_y(a,b)=0$ ((a,b) is a critical point of f).

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

① $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum

② $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum

③ $D<0 \rightarrow (a,b)$ is a saddle point

p809 Find the local maximum and minimum values and any saddle points

6. $f(x,y) = x^3y + 12x^2 - 8y$

$\rightarrow$ First find all critical points (solve $f_x=0$, $f_y=0$ for x and y)

$f_x = 3x^2y + 24x = 0 \rightarrow 3x(xy+8)=0$

$f_y = x^3 - 8 = 0 \rightarrow x^3 = 8 \rightarrow x = 2$

$3x(xy+8)=0$

$x^3=8 \to x=2$

Notice! Although $x=0$ makes $f_x=0$ it does NOT make $f_y=0$, so $x=0$ is not part of our critical points

$x=2 \xrightarrow{\text{Plug in to } f_x} 6(2y+8)=0 \quad y=-4$

Critical point $(2,-4)$

$D(2,-4)=f_{xx}(2,-4)f_{yy}(2,-4)-[f_{xy}(2,-4)]^2$

$\to$ we need to find f_{xx}, f_{yy}, f_{xy}

$f_{xx}=\frac{\partial}{\partial x}(3x^2y+24x)=24+6xy$

$f_{yy}=\frac{\partial}{\partial y}(x^3-8)=0$

$f_{xy}=3x^2 \qquad f_{xy}(2,-4)=12$

$D(2,-4)=(-24)(0)-[12]^2=-144<0$

$\boxed{(2,-4) \text{ is a saddle point}}$

8. $f(x,y)=e^{4y-x^2-y^2}$

$\to$ Find critical points: $f_x=(-2x)e^{4y-x^2-y^2}=0$

$f_y=(4-2y)e^{4y-x^2-y^2}=0$

$$(-2x)e^{4y-x^2-y^2} = 0$$

$$(4-2y)e^{4y-x^2-y^2} = 0$$

Notice: $e^{4y-x^2-y^2}$ is never $= 0$

so $f_x = 0$ if $x = 0$ and $f_y = 0$ if $y = 2$

Critical point is $(0,2)$

$f_{xx} = -2e^{4y-x^2-y^2} + (-2x)(-2x)e^{4y-x^2-y^2}$ (Using the Product Rule)

$f_{yy} = (-2)e^{4y-x^2-y^2} + (4-2y)(4-2y)e^{4y-x^2-y^2}$

$f_{xy} = (-2x)(4-2y)e^{4y-x^2-y^2}$

$f_{xx}(0,2) = -2e^{8-0^2-4} = -2e^4$

$f_{yy}(0,2) = -2e^4$

$f_{xy}(0,2) = 0$

$D(0,2) = (-2e^4)(-2e^4) - [0]^2 = 4e^8 > 0$

$D > 0 \quad f_{xx} < 0 \rightarrow$ $f(0,2)$ is a local max

$f(0,2) = e^4$