①

Mixing Problems

$$\frac{dx}{dt} = \underbrace{\underset{\text{flow rate in}}{\left(\frac{\text{volume}}{\text{time}}\right)}\underset{\text{concentration}}{\left(\frac{\text{amount}}{\text{volume}}\right)}}_{\text{input rate}} - \underbrace{\underset{\text{flow rate out}}{\left(\frac{\text{volume}}{\text{time}}\right)}\underset{\text{Concentration}}{\left(\frac{x(t)}{\text{Volume of fluid in tank}}\right)}}_{\text{output rate}}$$

From Advanced Engineering Mathematics 2nd Ed by Zill & Cullen p 80

23. A large tank is partially filled with 100 gallons of fluid in which 10 pounds of salt is dissolved. Brine containing $\frac{1}{2}$ lb of salt per gallon is pumped into the tank at a rate of 6 gal/min. The well-mixed solution is then pumped out at a slower rate of 4 gal/min. Find the number of pounds of salt in the tank after 30 minutes.

* First we find an equation for $\frac{dx}{dt}$ and solve it

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$$\frac{dx}{dt} = \left(\frac{6\,gal}{min}\right)\left(\frac{1/2\,lb}{gal}\right) - \left(\frac{4\,gal}{min}\right)\left(\frac{x(t)}{100+6t-4t}\right)$$

If the flow rate in $\neq$ flow rate out we need to put the volume at time t here
We start off with 100 gal and every minute add 6 gal and take away 4.

$$\frac{dx}{dt} = \left(\frac{6\,gal}{min}\right)\left(\frac{1/2\,lb}{gal}\right) - \left(\frac{4\,gal}{min}\right)\left(\frac{x(t)}{100+2t}\right)$$

$$\frac{dx}{dt} = 3 - \frac{4x}{100+2t}$$ (This is a linear 1st order d.e.)

→ Put it into standard form

$$\frac{dx}{dt} + \frac{4x}{100+2t} = 3$$

$$P(t) = \frac{4}{100+2t} = \frac{2}{50+t}$$

$$\mu(t) = e^{\int P(t)dt} = e^{\int \frac{2}{50+t}dt} = e^{2\ln(50+t)} = e^{\ln(50+t)^2}$$

$$\mu(t) = (50+t)^2$$

3 → multiply both sides of our d.e in Standard form by $\mu(t)$ to get

$$\underbrace{(50+t)^2 \frac{dx}{dt} + \frac{4x}{100+2t}(50+t)^2}_{\frac{d}{dt}\left[(50+t)^2 x\right]} = 3(50+t)^2$$

$$\frac{d}{dt}\left[(50+t)^2 x\right] = 3(50+t)^2$$

→ Integrate both sides

$$(50+t)^2 x = \int 3(50+t)^2 \, dt$$

$$(50+t)^2 x = \frac{\cancel{3}}{\cancel{3}}(50+t)^3 + C$$

$$x(t) = \frac{(50+t)^3 + C}{(50+t)^2}$$

$$x(0) = 10 \text{ lbs} = \frac{50^3 + C}{50^2} = 50 + \frac{C}{50^2}$$

$$-40 = \frac{C}{50^2} \qquad C = \underbrace{-40(50)^2}_{\text{You could leave C like this on a test}} = -100{,}000$$

$$\boxed{x(t) = \frac{(50+t)^3 - 100{,}000}{(50+t)^2} \; ; \; x(30) = 64.375 \text{ lbs}}$$

1) From *Calculus* by Munem & Foulis

P463

1. A tank initially contains 50 gallons of water into which 10 pounds of salt is dissolved. Pure water runs into the tank at the rate of 3 gal/min and is uniformly stirred into the solution. Meanwhile the mixture runs out of the tank at the constant rate of 2 gal/min. After how long is only 2 pounds of dissolved salt left in the tank?

$$\frac{dx}{dt} = \left(\frac{\text{volume}}{\text{time}}\right)\left(\frac{\text{amount}}{\text{volume}}\right) - \left(\frac{\text{volume}}{\text{time}}\right)\left(\frac{x(t)}{\text{volume of liquid in tank}}\right)$$

$$\frac{dx}{dt} = \left(\frac{3\text{gal}}{\text{min}}\right)\left(\frac{0\text{ lbs}}{\text{gal}}\right) - \left(\frac{2\text{gal}}{\text{min}}\right)\left(\frac{x(t)}{50+3t-2t}\right)$$

$$\frac{dx}{dt} = \frac{-2x}{50+t}$$

* This one is a separable d.e, even though flow rate in $\neq$ flow rate out

$$\int \frac{dx}{x} = \int \frac{-2}{50+t}\,dt$$

$$\ln|x| = -2\ln|50+t| + C$$

⑤ $\ln|x| = \ln(50+t)^{-2} + C$

$$|x| = e^{\ln(50+t)^{-2} + C}$$

$$x = Ke^{\ln(50+t)^{-2}} = K(50+t)^{-2}$$

$$x(0) = 10\text{lbs} = K(50+0)^{-2}$$

$$10 = \frac{K}{50^2}$$

$$K = 25{,}000$$

$$x(t) = 25000(50+t)^{-2}$$

* We have found an equation that gives the amount of salt in the tank at time t. Now we need to find how long it will take for $x(t) = 2$

$$2 = 25000(50+t)^{-2}$$

$$\frac{2}{25000} = \frac{1}{(50+t)^{-2}}$$

$$(50+t)^2 = \frac{25000}{2} = 12500$$

$$50+t = \sqrt{12500} \qquad t = \sqrt{12500} - 50 = \boxed{61.8 \text{ minutes}}$$

(6)

From Differential Equations & Their Applications

By Braun

P54

11. A 500 gallon tank originally contains 100 gallons of fresh water. Beginning at time $t=0$, water containing 50 percent pollutants flows into the tank at the rate of 2gal/min, and the well-stirred mixture leaves at the rate of 1gal/min. Find the concentration of pollutants in the tank the moment it overflows.

$$\frac{dx}{dt} = \underbrace{\left(\frac{\text{volume}}{\text{time}}\right)}_{\text{flow rate in}}\underbrace{\left(\frac{\text{amount}}{\text{volume}}\right)}_{\text{concentration}} - \underbrace{\left(\frac{\text{volume}}{\text{time}}\right)}_{\text{flow rate out}}\underbrace{\left(\frac{x(t)}{\text{volume of liquid}}\right)}_{\text{concentration}}$$

$$\frac{dx}{dt} = \left(\frac{2\text{gal}}{\text{min}}\right)(.50) - \left(\frac{1\text{gal}}{\text{min}}\right)\left(\frac{x(t)}{100+2t-1t}\right)$$

50% pollutants
→ concentration of pollutants is .5

$$\frac{dx}{dt} = 1 - \frac{x}{100+t}$$

$$\frac{dx}{dt} + \frac{x}{100+t} = 1$$

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$$P(t) = \frac{1}{100+t}$$

$$\mu(t) = e^{\int P(t)\,dt} = e^{\int \frac{1}{100+t}\,dt} = e^{\ln(100+t)}$$

$$\mu(t) = 100 + t$$

$$\underbrace{(100+t)\frac{dx}{dt} + x\left(\frac{100+t}{100+t}\right)}_{\frac{d}{dt}\left[(100+t)x\right]} = 1(100+t)$$

$$\frac{d}{dt}\left[(100+t)x\right] = 100+t$$

$$(100+t)x = \int 100 + t \; dt$$

$$(100+t)x = 100t + \tfrac{1}{2}t^2 + C$$

$$x(t) = \frac{100t + \frac{1}{2}t^2 + C}{100+t}$$

$x(0) = 0$ no pollutants in water to start

$$x(0) = 0 = \frac{C}{100} \rightarrow C = 0$$

$$x(t) = \frac{100t + \frac{1}{2}t^2}{100+t}$$

Tank overflows when $500 = 100 + 2t - t$

$400 = t$

$$x(400) = \frac{100(400) + \frac{1}{2}(400)^2}{500} = 240$$

(8) But we want the concentration of pollutents when the tank overflows

so

$X(400) = 240$

$$\frac{240}{\text{volume at time of overflow}} = \frac{240}{500} = .48$$

$\boxed{\text{so 48\% of the water contains pollutents}}$