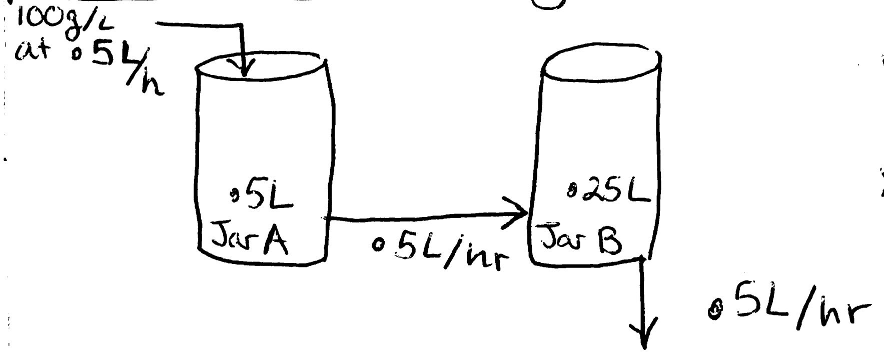
Mixing Problems with Interconnected Tanks

* These problems build on our knowledge of mixing problems with a single tank and our recent work with matrices.

Examples from Introduction to Differential Equations by Campbell & Haberman p377-379

Ex 1) Two jars are initially full of pure water. Jar A is 0.5L and Jar B is 0.25L. Water containing salt at a concentration of 100 g/L is pumped into jar A at 0.5L/h. Water flows from jar A to jar B at 0.5L/h. Water flows out of jar B and down a drain at 0.5 L/h. Find the amount of salt in each jar at time t.



Initial conditions:

$X_1(0)=0$
$X_2(0)=0$
They told us we started with pure water

Let $X_1(t)$ be the amount of salt in jar A at time t
$X_2(t)$ be the amount of salt in jar B at time t

Then $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$ represent how the amount of salt in jar A and jar B change as time passes.

$$\frac{dx_1}{dt} = \text{Input rate} - \text{Output rate}$$

$$= \begin{pmatrix}\text{Concentration}\\ \text{in}\end{pmatrix}\begin{pmatrix}\text{flow rate}\\ \text{in}\end{pmatrix} - \begin{pmatrix}\text{Concentration}\\ \text{out}\end{pmatrix}\begin{pmatrix}\text{flow}\\ \text{rate}\\ \text{out}\end{pmatrix}$$

$$= \left(\frac{100g}{L}\right)\left(\frac{.5L}{hr}\right) - \underbrace{\left(\frac{x_1}{.5L}\right)}\left(\frac{.5L}{hr}\right)$$

Concentration out is the amount of salt in jar A divided by the amount of liquid in jar A. $x_1(t)$ = salt in jar A and .5L is the amount of liquid

$$\frac{dx_2}{dt} = \left(\frac{x_1}{.5L}\right)\left(\frac{.5L}{hr}\right) - \underbrace{\left(\frac{x_2}{.25L}\right)}\left(\frac{.5L}{hr}\right)$$

x_2 = amount of salt in jar B
.25 = Volume of jar B

Simplifying: $\frac{dx_1}{dt} = 50 - x_1$

$$\frac{dx_2}{dt} = x_1 - 2x_2$$

* We can represent this using matrices -- this is useful since we can solve this using our matrix techniques

P3

We want to represent it using

$$\vec{x}\,' = A\vec{x} + \vec{f} \qquad \vec{x}(0) = \vec{x_0}$$

Our system of equations has 2 derivatives, $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$; these will become our $\vec{x}\,'$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix}$$

Once we have our $\vec{x}\,'$, we automatically have our $\vec{x}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \vec{f}$$

A is a matrix that contains the coefficients of x_1 and x_2

* For $\frac{dx_1}{dt} = 50 - x_1$ the coefficient of x_1 is -1, the coefficient of x_2 is 0 (because there isn't an x_2)
Whatever is left over belongs to $\vec{f}$, in this case it is 50.

* For $\frac{dx_2}{dt} = x_1 - 2x_2$ the coefficient of x_1 is 1 and for x_2 it is -2, and there isn't anything leftover for $\vec{f}$

So our $A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$ and $\vec{f} = \begin{pmatrix} 50 \\ 0 \end{pmatrix}$

* You'll know that you found the correct values of A and $\vec{f}$ if when you multiply everything out you get our original system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 50 \\ 0 \end{pmatrix}$$

Double checking:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} (-1x_1 + 0x_2) \\ (1x_1 - 2x_2) \end{pmatrix} + \begin{pmatrix} 50 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -x_1 + 50 \\ x_1 - 2x_2 \end{pmatrix} \quad ✓$$

* For a test you will just set up the problem so your final answer would be:

$$\boxed{\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' &= \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 50 \\ 0 \end{pmatrix} \\ \vec{x}(0) &= \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}}$$

* This problem looks like it might be pleasant so let's work it out for extra practice

P5

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$$

* 1st we need $\vec{X_c}$ and to find that we need the eigenvalues and eigenvectors of A

Eigenvalues: $|A - rI| = 0$

$$\begin{vmatrix} -1-r & 0 \\ 1 & -2-r \end{vmatrix} = (-1-r)(-2-r) - 0(1) = 0$$

or $(-1-r)(-2-r) = 0$

$\boxed{r_1 = -1 \quad r_2 = -2 \quad \text{are our eigenvalues}}$

Eigenvectors: $(A - rI)\vec{u} = \vec{0}$

$r_1 = -1$: $(A - (-1)I)\vec{u} = \vec{0}$

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$u_1 - u_2 = 0 \quad u_1 = u_2$ $\boxed{\vec{u_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}s \quad \text{1st eigenvector}}$

$r_2 = -2$: $(A - (-2)I)\vec{u} = \vec{0}$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $u_1 = 0$, u_2 has no restrictions

so $\vec{u_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}s$

$$\boxed{\vec{X_c} = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

* Now we need $\vec{X_p}$. Let's find it using our Variation of Parameters method

$$\vec{X_p} = \Phi \int \Phi^{-1} \vec{f}$$

* We need Φ the fundamental matrix; remember we get that from $\vec{X_c}$

$$\vec{X_c} = c_1 e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\text{so } \Phi = \begin{pmatrix} 1e^{-t} & 0e^{-2t} \\ 1e^{-t} & 1e^{-2t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ e^{-t} & e^{-2t} \end{pmatrix}$$

* Let's use our formula for the inverse of a 2x2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

when A^{-1} exists

$$\text{so } \Phi^{-1} = \frac{1}{e^{-t}e^{-2t} - 0e^{-t}}\begin{pmatrix} e^{-2t} & 0 \\ -e^{-t} & e^{-t} \end{pmatrix}$$

$$= e^{3t}\begin{pmatrix} e^{-2t} & 0 \\ -e^{-t} & e^{-t} \end{pmatrix} = \begin{pmatrix} e^{t} & 0 \\ -e^{2t} & e^{2t} \end{pmatrix}$$

$$\vec{X_p} = \Phi \int \Phi^{-1}\vec{f} = \Phi \int \begin{pmatrix} e^{t} & 0 \\ -e^{2t} & e^{2t} \end{pmatrix}\begin{pmatrix} 50 \\ 0 \end{pmatrix}$$

$$= \Phi \int \begin{pmatrix} 50e^{t} \\ -50e^{2t} \end{pmatrix} dt = \begin{pmatrix} e^{-t} & 0 \\ e^{-t} & e^{-2t} \end{pmatrix}\begin{pmatrix} 50e^{t} \\ -\frac{50}{2}e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 50 \\ 50-25 \end{pmatrix} = \begin{pmatrix} 50 \\ 25 \end{pmatrix}$$

So our general solution is

$$\vec{X} = \vec{X}_c + \vec{X}_p$$

$$\boxed{\vec{X} = c_1 e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}50\\25\end{pmatrix}}$$

* But we have initial conditions

$\vec{X}(0) = \begin{pmatrix}0\\0\end{pmatrix}$. so we can use the general solution to solve for c_1 and c_2.

$$\vec{X}(0) = c_1 e^{-0}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-2(0)}\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}50\\25\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}50\\25\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$c_1 + 50 = 0 \quad c_1 = -50$$

$$c_1 + c_2 + 25 = 0 \rightarrow c_2 = 25$$

Final answer!

$$\vec{X} = -50e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + 25e^{-2t}\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}50\\25\end{pmatrix}$$

or

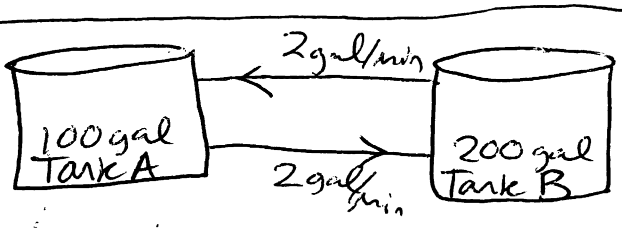
$$\vec{X} = \begin{pmatrix}-50e^{-t} + 50\\ -50e^{-t} + 25e^{-2t} + 25\end{pmatrix}$$

So $\boxed{x_1(t) = -50e^{-t} + 50 \text{ and } x_2(t) = -50e^{-t} + 25e^{-2t} + 25}$

Note! as $t \rightarrow \infty$ $x_1 \rightarrow 50$ g of salt and $x_2 \rightarrow 25$ g of salt

Ex 2 There are 2 tanks. Tank A is a 100-gal tank, initially full of water containing salt at a concentration of 0.5 lb/gal. Tank B is a 200-gal tank, initially full of water containing salt at a concentration of .1 lb/gal. Starting at time $t=0$, water is pumped from tank A to tank B at 2 gal/min and from tank B to tank A at 2 gal/min. If $x_1(t)$ is the amount of salt in tank A and $x_2(t)$ is the amount of salt in tank B at time t. Find A, $\vec{f}$, and $\vec{x_0}$ so that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \vec{f} \qquad \vec{x}(0) = \vec{x_0}$$



$$\text{Concentration} = \frac{\text{amount}}{\text{volume}} = \frac{x_1(0)}{100} = .5\,\text{lb/gal}$$

$$x_1(0) = 50\,\text{lb}$$

$$\text{Concentration} = \frac{x_2(0)}{200} = .1\,\text{lb/gal}$$

$$x_2(0) = 20\,\text{lb}$$

$$\frac{dx_1}{dt} = \left(\frac{x_2}{200}\right)\left(\frac{2\,\text{gal}}{\text{min}}\right) - \left(\frac{x_1}{100}\right)\left(\frac{2\,\text{gal}}{\text{min}}\right)$$

$$\frac{dx_1}{dt} = \frac{x_2}{100} - \frac{x_1}{50}$$

$$\frac{dx_2}{dt} = \left(\frac{x_1}{100}\right)\left(\frac{2\,gal}{min}\right) - \left(\frac{x_2}{200}\right)\left(\frac{2\,gal}{min}\right)$$

$$\frac{dx_2}{dt} = \frac{x_1}{50} - \frac{x_2}{100}$$

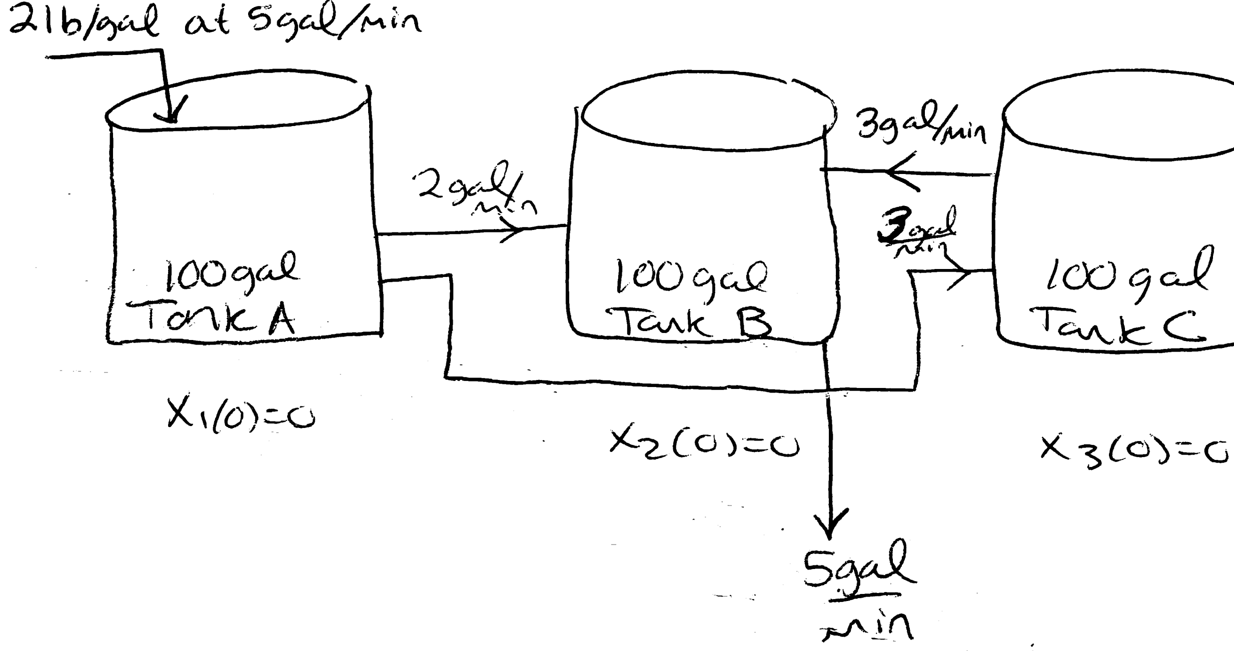
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} \frac{-1}{50} & \frac{1}{100} \\ \frac{-1}{100} & \frac{1}{50} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\vec{f}}$$

$$\vec{x}(0) = \begin{pmatrix} 50 \\ 20 \end{pmatrix}$$

Ex 3 There are 3 tanks, each initially containing 100 gal of pure water. Water containing salt at a concentration of 2 lb/gal flows into tank A at a rate of 5 gal/min. Water is pumped from tank A to tank B at 2 gal/min and from tank A to tank C at 3 gal/min. Water is pumped from tank C to tank B at 3 gal/min. Water is pumped out of tank B and then down a drain at a rate of 5 gal/min.

If $x_1(t)$, $x_2(t)$, and $x_3(t)$ are the amounts of salt in tank A, tank B, and tank C respectively, find A, $\vec{f}$, and $\vec{x}_0$ so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \vec{f} \qquad \vec{x}(0) = \vec{x}_0$$



$$\frac{dx_1}{dt} = \left(2\,\frac{\text{lb}}{\text{gal}}\right)\left(\frac{5\,\text{gal}}{\text{min}}\right) - \left(\frac{x_1}{100\,\text{gal}}\right)\left(\frac{2\,\text{gal}}{\text{min}}\right) - \left(\frac{x_1}{100\,\text{gal}}\right)\left(\frac{3\,\text{gal}}{\text{min}}\right)$$

$$\frac{dx_1}{dt} = 10 - \frac{5x_1}{100} = 10 - \frac{x_1}{20}$$

$$\frac{dx_2}{dt} = \left(\frac{x_1}{100\,\text{gal}}\right)\left(\frac{2\,\text{gal}}{\text{min}}\right) + \left(\frac{x_3}{100\,\text{gal}}\right)\left(\frac{3\,\text{gal}}{\text{min}}\right) - \left(\frac{x_2}{100\,\text{gal}}\right)\left(\frac{5\,\text{gal}}{\text{min}}\right)$$

$$\frac{dx_2}{dt} = \frac{x_1}{50} + \frac{3x_3}{100} - \frac{x_2}{20}$$

$$\frac{dx_3}{dt} = \left(\frac{x_1}{100\,\text{gal}}\right)\left(\frac{3\,\text{gal}}{\text{min}}\right) - \left(\frac{x_3}{100\,\text{gal}}\right)\left(\frac{3\,\text{gal}}{\text{min}}\right)$$

$$\frac{dx_3}{dt} = \frac{3x_1}{100} - \frac{3x_3}{100}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} \frac{-1}{20} & 0 & 0 \\ \frac{1}{50} & \frac{-1}{20} & \frac{3}{100} \\ \frac{3}{100} & 0 & \frac{-3}{100} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\vec{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad A = \begin{pmatrix} \frac{-1}{20} & 0 & 0 \\ \frac{1}{50} & \frac{-1}{20} & \frac{3}{100} \\ \frac{3}{100} & 0 & \frac{-3}{100} \end{pmatrix} \qquad \vec{f} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}}$$