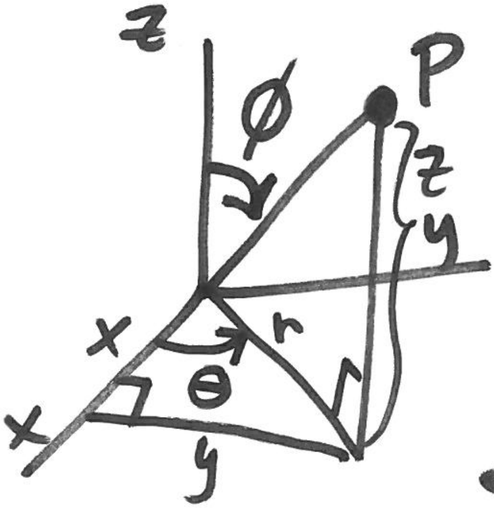
More Help with Spherical

Recall: $x = \rho \sin\phi \cos\theta$



$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$x^2 + y^2 + z^2 = \rho^2$

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

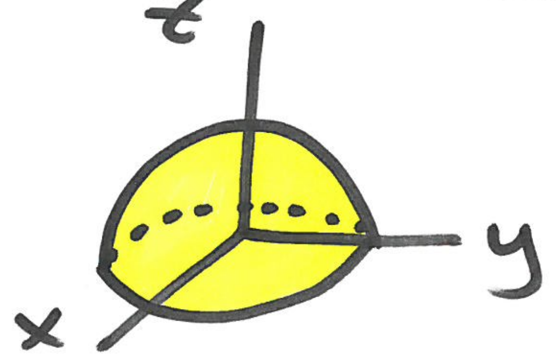
θ = same as polar & cylindrical, angle between the positive x-axis. Find this by projecting F onto the xy-plane

ϕ = angle between the positive z-axis & F.

Sometimes helpful to draw the projection of F in the xz or yz-planes

ρ = distance from origin to F

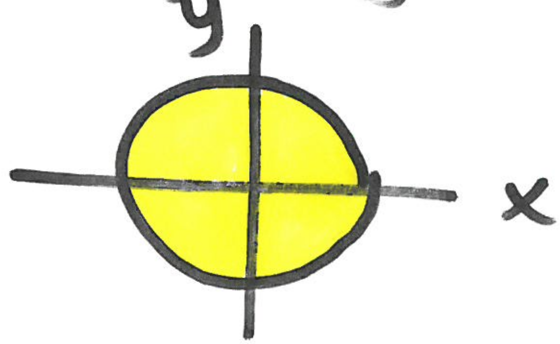
Ex 1 F is bounded by $z=\sqrt{25-x^2-y^2}$ and the xy-plane



$x^2+y^2+z^2=25$

so $\boxed{0 \le \rho \le 5}$

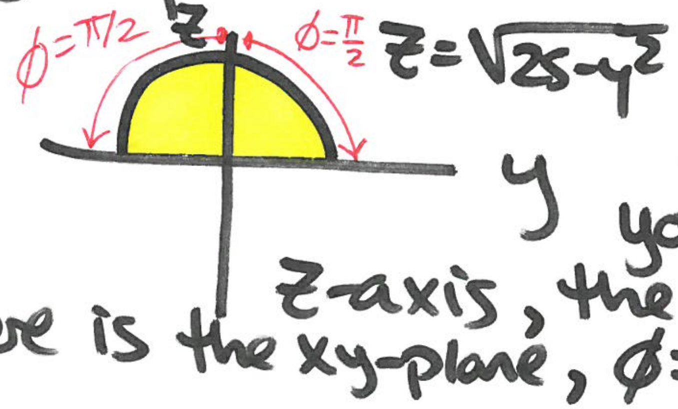
Find θ by projecting this onto the xy plane



Whole circle centered at the origin

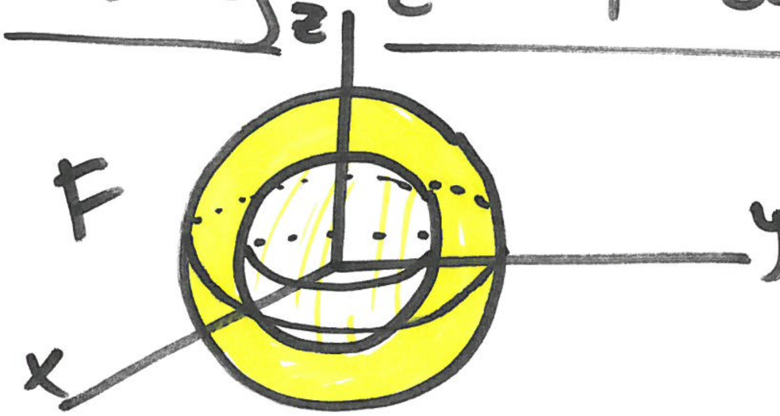
$\boxed{0 \le \theta \le 2\pi}$

Find ϕ by projecting F onto the yz-plane



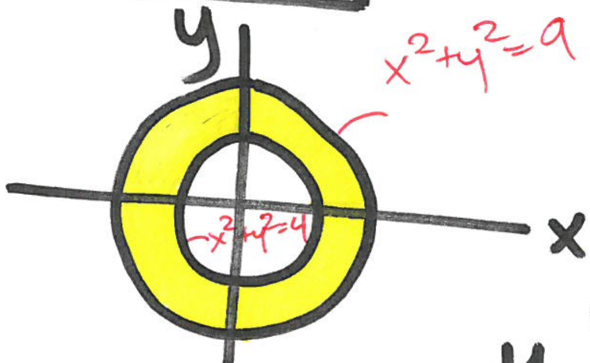
F is touching the positive z-axis so ϕ starts at 0. Imagine your bicep is the positive z-axis, the furthest we drop down ere is the xy-plane, $\phi=\pi/2$. $\boxed{0 \le \phi \le \pi/2}$

Ex 2 F is the solid between $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$

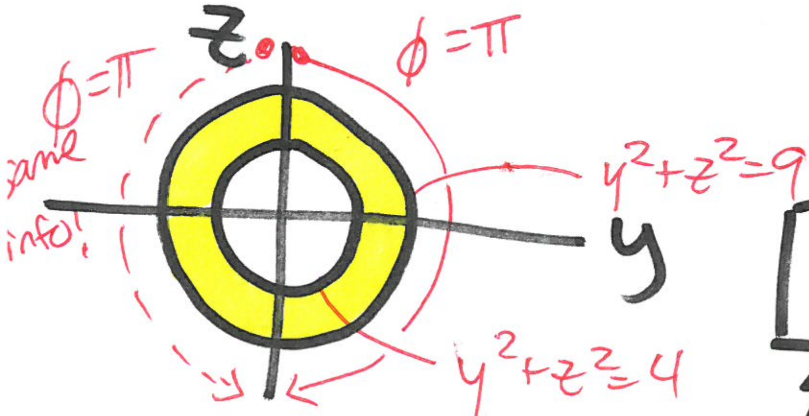


Hard to color in; imagine the Earth's mantle.

$\boxed{2 \le \rho \le 3}$



Project F on the xy-plane. Whole circle centered at the origin $\boxed{0 \le \theta \le 2\pi}$

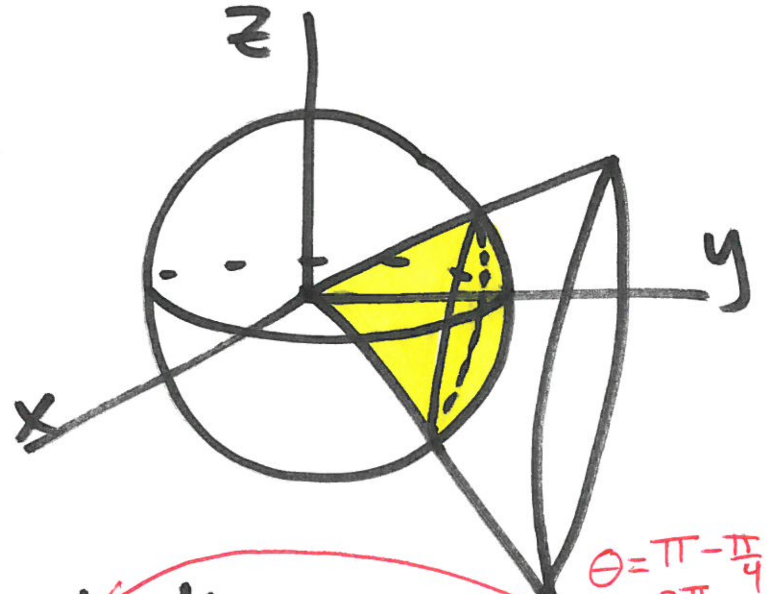


$\boxed{0 \le \phi \le \pi}$

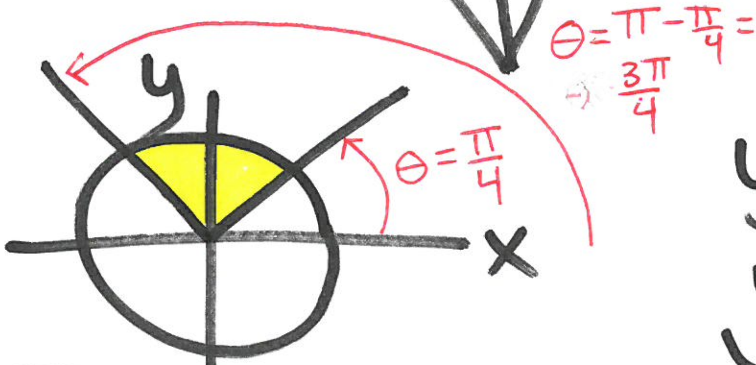
touching positive z-axis

touching negative z-axis

Ex 3a F is bounded by $x^2+y^2+z^2=1$ & inside the cone $y=\sqrt{x^2+z^2}$



$$\boxed{0 \le \rho \le 1}$$



$$y=\sqrt{x^2+0^2}=\sqrt{x^2}$$

$\underbrace{y=x}_{\text{bisects 1st quadrant}}$, $y=-x$ where $y \ge 0$

$$\boxed{\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}}$$

This problem is actually kind of tricky (definitely tricky).

The angle ϕ is what makes this Challenging. Since our cone is $y=\sqrt{x^2+z^2}$, the angle from the positive z-axis to the outside of the cone is Changing wrt θ.

$$y=\sqrt{x^2+z^2} \longrightarrow \text{square both sides}$$

$$y^2=x^2+z^2$$

$$\cancel{\rho^2}\sin^2\phi\sin^2\theta=\cancel{\rho^2}\sin^2\phi\cos^2\theta+\cancel{\rho^2}\cos^2\phi$$

$$\sin^2\phi(\sin^2\theta-\cos^2\theta)=\cos^2\phi$$

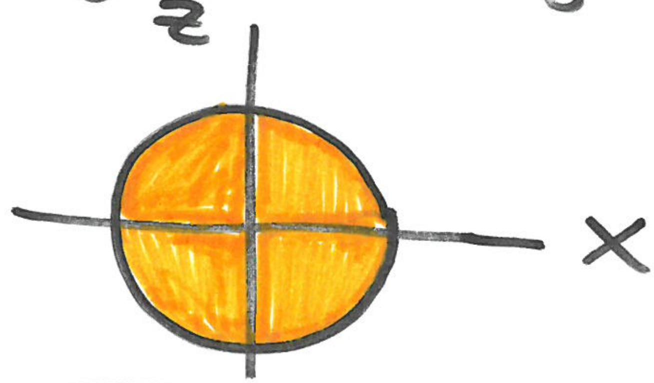
$$\frac{\sin^2\phi}{\cos^2\phi}=\frac{1}{\sin^2\theta-\cos^2\theta}$$

$$\tan\phi=\frac{1}{\sqrt{\sin^2\theta-\cos^2\theta}}$$

$$\phi=\tan^{-1}\left(\frac{1}{\sqrt{\sin^2\theta-\cos^2\theta}}\right)$$

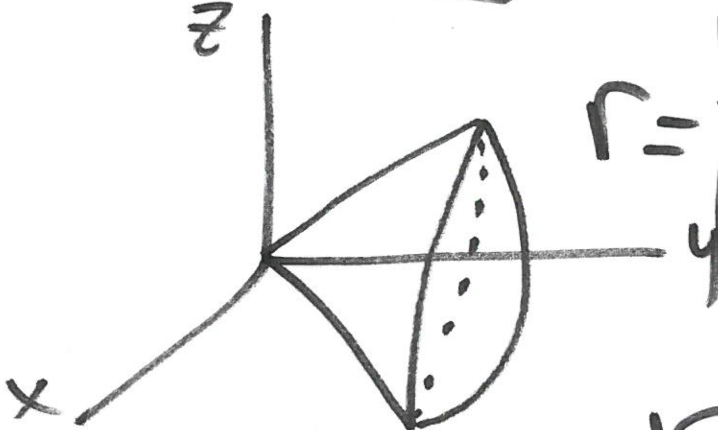
$$\tan^{-1}\left(\frac{1}{\sqrt{\sin^2\theta-\cos^2\theta}}\right)\le\phi\le\pi-\tan^{-1}\left(\frac{1}{\sqrt{\sin^2\theta-\cos^2\theta}}\right)$$

Ex 3b Same problem in augmented cylindrical coordinates



$$x = r\cos\theta$$
$$z = r\sin\theta$$
$$x^2 + z^2 = r^2$$
$$dV = r\,dy\,dr\,d\theta$$

$$\boxed{0 \le \theta \le 2\pi}$$



$$\boxed{r = \underbrace{\sqrt{r^2}}_{\sqrt{x^2+z^2}} \le y \le \sqrt{1-r^2}}$$

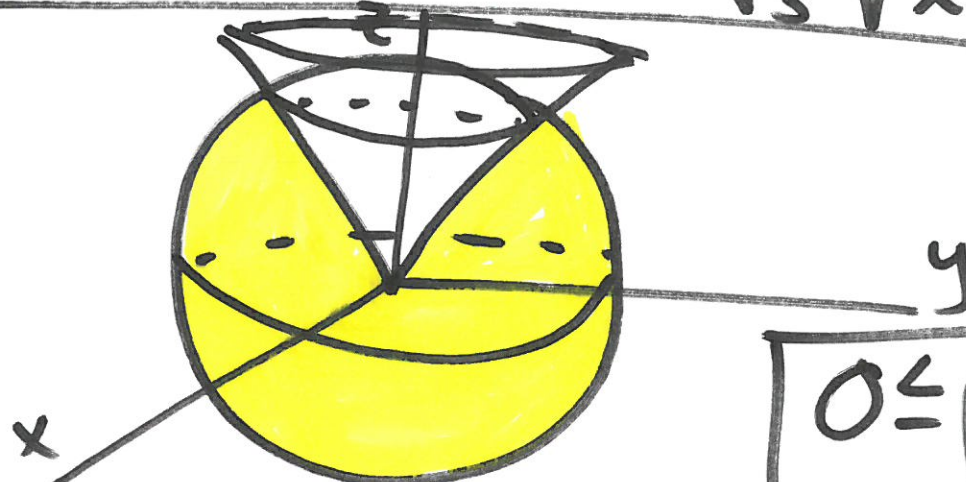
left surface = right surface

$$\sqrt{r^2} = \sqrt{1-r^2}$$
$$r^2 = 1 - r^2$$
$$1 = 2r^2$$
$$\frac{1}{2} = r^2$$

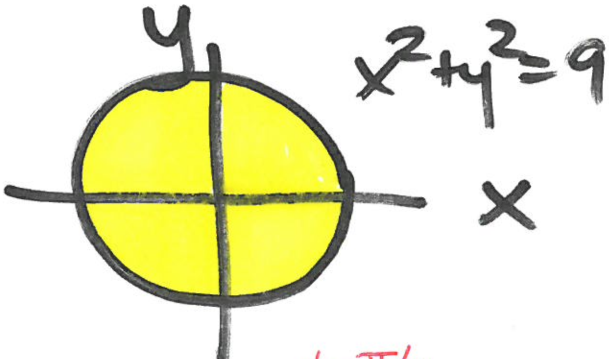
$$\boxed{0 \le r \le \frac{1}{\sqrt{2}}}$$

Note:
It is much easier this way

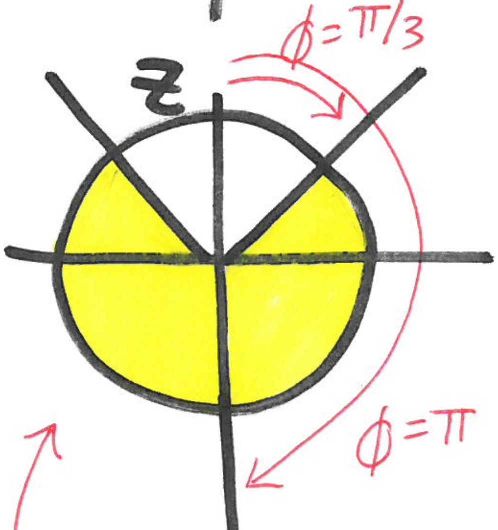
Ex 4 F is the solid within the sphere $x^2+y^2+z^2=9$ & below the cone $z=\frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$



$$\boxed{0 \le \rho \le 3}$$



$$\boxed{0 \le \theta \le 2\pi}$$



both sides of the z-axis yield the same info

$$z=\frac{1}{\sqrt{3}}\sqrt{x^2+y^2}=\frac{1}{\sqrt{3}}\sqrt{r^2}$$

$$\cancel{\rho}\cos\phi=\frac{1}{\sqrt{3}}r=\frac{1}{\sqrt{3}}\cancel{\rho}\sin\phi$$

$$\sqrt{3}=\tan\phi$$

$$\phi=\tan^{-1}\sqrt{3}=\pi/3$$

$$(\tan \pi/3=\sqrt{3})$$

$$\boxed{\frac{\pi}{3} \le \phi \le \pi}$$

Ex 4 cont.

Or $$z = \frac{1}{\sqrt{3}}\sqrt{0^2+y^2}$$

$$z = \frac{1}{\sqrt{3}} y$$

$$\cancel{\rho}\cos\phi = \frac{1}{\sqrt{3}}\cancel{\rho}\sin\phi\sin\theta$$

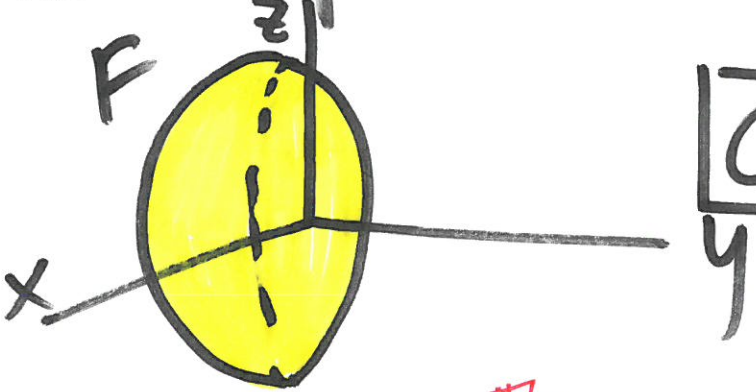
but $\theta = \pi/2$ since we are in the yz-plane directly above the y-axis

$$\cos\phi = \frac{1}{\sqrt{3}}\sin\phi \underbrace{\cancel{\sin\pi/2}}_{1}$$

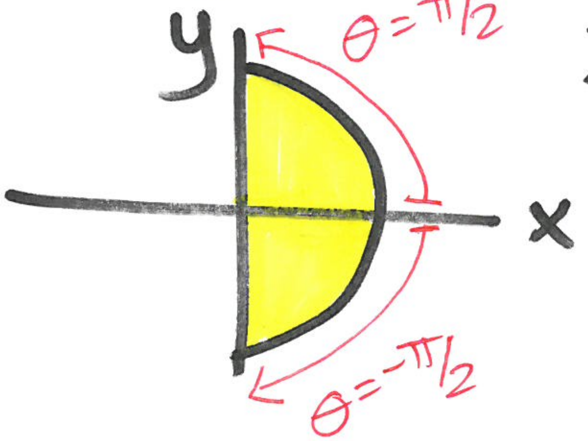
$$\tan\phi = \sqrt{3}$$

1st way for determing ϕ makes more sense to me.

Ex 5 F is the solid bounded by the yz-plane & the hemisphere $x=\sqrt{9-y^2-z^2}$

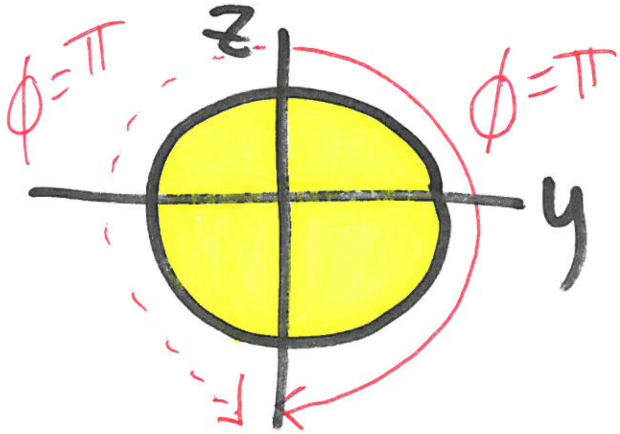


$$\boxed{0 \le \rho \le 3}$$



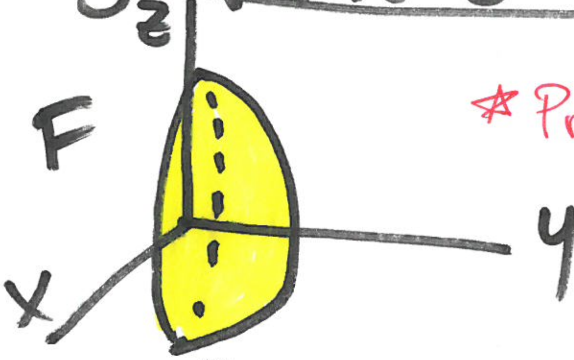
$$x=\sqrt{9-y^2-0^2}$$

$$\boxed{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$$



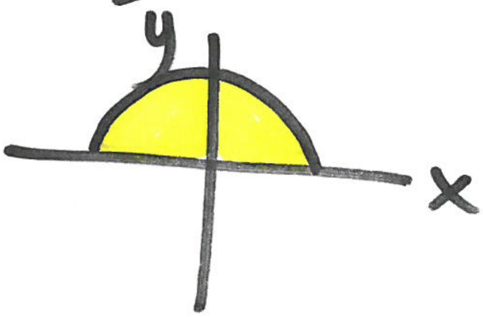
$$\boxed{0 \le \phi \le \pi}$$

Ex 6a F is bounded by $y=\sqrt{4-x^2-z^2}$ & the xz-plane



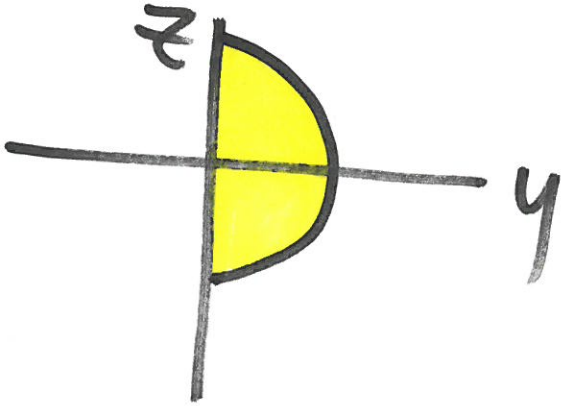
* Pretend this looks like a hemisphere

$$\boxed{0\le\rho\le 2}$$



$$y=\sqrt{4-x^2-0^2}$$

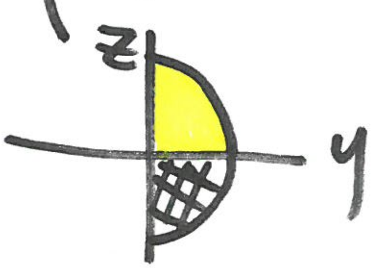
$$\boxed{0\le\theta\le\pi}$$



$$\boxed{0\le\phi\le\pi}$$

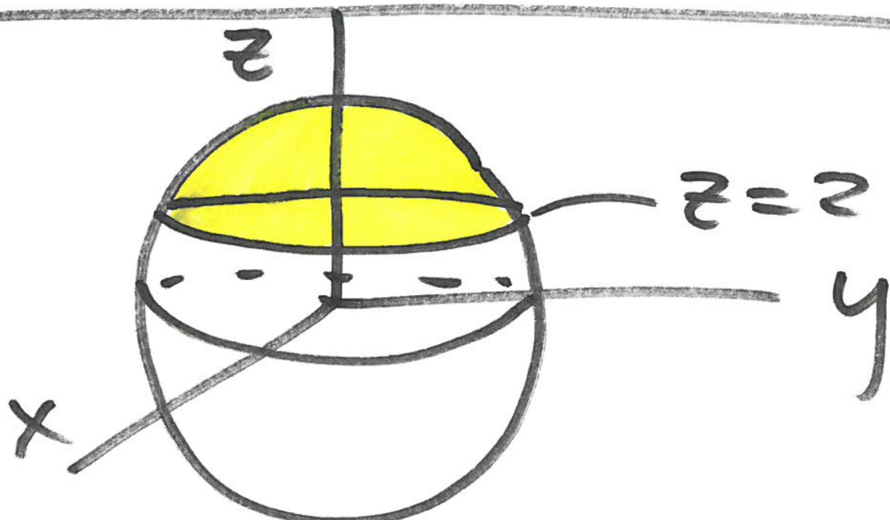
Ex 6b F bounded by $y=\sqrt{4-x^2-z^2}$, xz-plane, & above the xy plane

$0\le\rho\le 2$ $\quad$ $0\le\theta\le\pi$



$$0\le\phi\le\frac{\pi}{2}$$

Ex 7 F is bounded above by $x^2+y^2+z^2=9$ & below by $z=2$



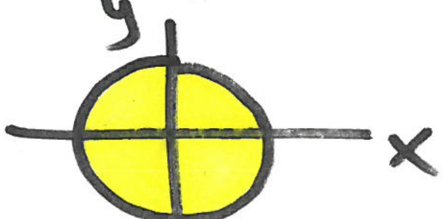
FYI: Probably better in cylindrical.

$z=2$

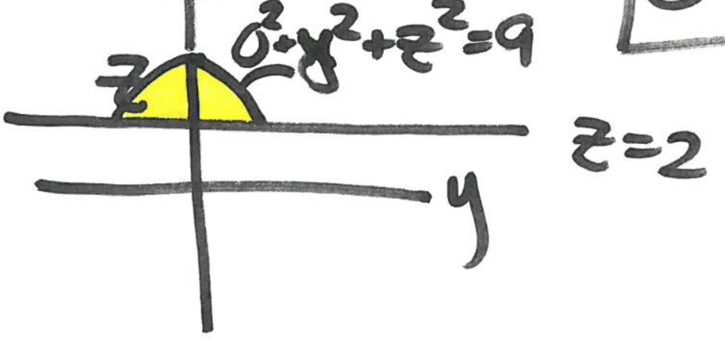
$\rho\cos\phi=2$

$\rho = 2/\cos\phi$

$$\boxed{\frac{2}{\cos\phi} \le \rho \le 3}$$



$$\boxed{0 \le \theta \le 2\pi}$$



$y^2+2^2=9$

$y^2=5 \rightarrow y=\sqrt{5}$

$\rho\sin\phi\sin\theta=\sqrt{5}$

Ex 7 cont!

$$\underbrace{\rho \sin\phi \sin\theta}_{y} = \sqrt{5}$$

$$3 \sin\phi \sin \pi/2 = \sqrt{5}$$

ρ where $z=2$

θ in yz-plane above y-axis

$$\sin\phi = \frac{\sqrt{5}}{3}$$

$$\phi = \sin^{-1}\left(\frac{\sqrt{5}}{3}\right)$$

$$\boxed{0 \le \phi \le \sin^{-1}\left(\frac{\sqrt{5}}{3}\right)}$$