11.3 Partial Derivatives

- To find f_x treat y as a constant & differentiate with respect to x
- To find f_y treat x as a constant & differentiate with respect to y

Calc 1 $\frac{d}{dx}(a) = 0$ $\quad \frac{d}{dx}[af(x)] = af'(x)$

Ex 1: Find $\frac{\partial z}{\partial x}$ if $z = 12 + x^2 + y^2 + xy$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(12) + \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(xy)$$
$$= 0 + 2x + 0 + 1 \cdot y$$
$$= \boxed{2x + y}$$

as far as x is concerned y is a constant, so $\frac{\partial}{\partial x}(y^2) = 0$

Calc 1 Chain Rule

$$\frac{d}{dx}(f(g(x))) = f'(g(x))\, g'(x)$$

$$\frac{d}{dx}[\ln(f(x))] = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

Ex 2: Find f_x and f_y if $f(x,y) = \ln(x^3 + xy^2 + 3y)$

$$f_x = \frac{1}{x^3 + xy^2 + 3y} \cdot \frac{\partial}{\partial x}(x^3 + xy^2 + 3y) = \boxed{\frac{1}{x^3 + xy^2 + 3y}(3x^2 + y^2)}$$

$$f_y = \frac{1}{x^3+xy^2+3y} \frac{\partial}{\partial y}\left(x^3+xy^2+3y\right)$$

Notice the similarities between what we're doing and the Calc 1 technique.

$$f_y = \frac{1}{x^3+xy^2+3y}\left(\frac{\partial}{\partial y}(x^3) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial y}(3y)\right)$$

$$= \frac{1}{x^3+xy^2+3y}\left(0 + 2xy + 3\right)$$

x is constant w.r.t y

when multiplying the other variable comes along for the ride, like in $\frac{d}{dx}[af(x)] = af'(x)$

$$f_y = \boxed{\frac{2xy+3}{x^3+xy^2+3y}}$$

$\boxed{\text{Calc 1}}$ More Chain Rule

$$\frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)}f'(x)$$

$$\frac{d}{dx}\left[\cos(f(x))\right] = -\sin(f(x))\,f'(x)$$

$$\frac{d}{dx}\left[\tan(f(x))\right] = \sec^2(f(x))\,f'(x)$$

$$\frac{d}{dx}\left[(f(x))^n\right] = n(f(x))^{n-1}f'(x)$$

These examples give us more practice with the Chain Rule (from Calc1)

Ex 3: Find $\frac{\partial w}{\partial t}$ if $w = e^{-s^2t^3}$

$$\frac{\partial w}{\partial t} = e^{-s^2t^3}\frac{\partial}{\partial t}(-s^2t^3) = \boxed{e^{-s^2t^3}(-3s^2t^2)}$$

For practice: $\frac{\partial w}{\partial s} = e^{-s^2t^3}(-2st^3)$

Ex 4: $f(x,y) = \tan(x^5+2y)$

$f_x = \boxed{5x^4\sec^2(x^5+2y)}$ $\quad f_y = \boxed{2\sec^2(x^5+2y)}$

Ex 5: $f(x,y,z) = \sqrt{x^2+yz+z^2}$

Find f_x, f_y & f_z

- First rewrite f, $f = (x^2+yz+z^2)^{1/2}$

$$f_x = \tfrac{1}{2}(x^2+yz+z^2)^{-1/2}\frac{\partial}{\partial x}(x^2+yz+z^2)$$
$$= \boxed{\tfrac{1}{2}(x^2+yz+z^2)^{-1/2}(2x)}$$

$$f_y = \tfrac{1}{2}(x^2+yz+z^2)^{-1/2}\frac{\partial}{\partial y}(x^2+yz+z^2)$$
$$= \boxed{\tfrac{1}{2}(x^2+yz+z^2)^{-1/2}(z)}$$

$$f_z = \tfrac{1}{2}(x^2+yz+z^2)^{-1/2}\frac{\partial}{\partial z}(x^2+yz+z^2)$$
$$= \boxed{\tfrac{1}{2}(x^2+yz+z^2)^{-1/2}(y+2z)}$$

Calc 1 Product Rule

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

Ex 6: $f(x,y) = \sqrt{x}\, y \sin(2x+5y)$

Let's start with f_y since it is easier

$$f_y = \frac{\partial}{\partial y}(\sqrt{x}\, y)\sin(2x+5y) + (\sqrt{x}\, y)\frac{\partial}{\partial y}(\sin(2x+5y))$$

"derivative of the 1st times the 2nd plus the 1st times the derivative of the 2nd"

$$f_y = \boxed{\sqrt{x}\sin(2x+5y) + \sqrt{x}\, y\cos(2x+5y)\cdot \underbrace{5}_{\frac{\partial}{\partial y}(2x+5y)}}$$

$$f_x = \frac{\partial}{\partial x}(\sqrt{x}\, y)\sin(2x+5y) + (\sqrt{x}\, y)\frac{\partial}{\partial x}(\sin(2x+5y))$$

rewrite as $x^{1/2}$

$$= \frac{\partial}{\partial x}(x^{1/2}y)\sin(2x+5y) + (\sqrt{x}\, y)\frac{\partial}{\partial x}(\sin(2x+5y))$$

$$= \boxed{\frac{1}{2}x^{-1/2}y\sin(2x+5y) + \sqrt{x}\, y\cos(2x+5y)\, \underbrace{2}_{\frac{\partial}{\partial x}(2x+5y)}}$$

Calc 1 Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Ex 7: $f(x,y) = \dfrac{\cos(x+y^2)}{3xy^3}$

$$f_x = \frac{\frac{\partial}{\partial x}\left[\cos(x+y^2)\right](3xy^3) - \left[\cos(x+y^2)\right]\frac{\partial}{\partial x}(3xy^3)}{(3xy^3)^2}$$

$$f_x = \boxed{\frac{-\sin(x+y^2)\,(3xy^3) - \cos(x+y^2)\,(3y^3)}{(3xy^3)^2}}$$

Miscellaneous Examples

Ex 8: $z = \dfrac{x}{y^2}$ Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

Rewrite z, $z = xy^{-2}$

$\dfrac{\partial z}{\partial x} = \boxed{y^{-2}}$ $\dfrac{\partial z}{\partial y} = \boxed{-2xy^{-3}}$

Ex 9: $f(x,y) = e^{\frac{x}{y^2}}$ Find f_x and f_y

Rewrite f, $f = e^{xy^{-2}}$

$f_x = \boxed{e^{xy^{-2}}\, y^{-2}}$ $f_y = \boxed{e^{xy^{-2}}(-2xy^{-3})}$