Power Series

Def: $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$
is called a power series centered at a.

Technique: To find the radius of convergence & the interval of convergence

1. Use the Ratio Test to find R = radius of convergence
2. Check the convergence of the series at endpoints $x=a-R$, $x=a+R$
3. If the series converges at an endpoint include it in your interval of convergence

Examples from Calculus: Early Transcendental Functions By Larson, Hostetler, Edwards

→ Find the radius & interval of convergence

$\boxed{\text{EX 1}}$ $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{\left(\frac{x}{2}\right)^{n+1}}{\left(\frac{x}{2}\right)^n} \right| = \lim_{n\to\infty} \left|\frac{x}{2}\right| = \left|\frac{x}{2}\right| < 1$$

converges when < 1

→ $|x| < 2$

* Once we get it into the form $|x-a| < R$, R = Radius of convergence *

$R = 2$

End points $\quad X = a - R \longrightarrow X = 0 - 2 = -2$

$X = a + R \qquad X = 0 + 2 = 2$

$X = -2$ $\quad$ * Recall: Ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} L<1 & \text{converges} \\ L>1 & \text{diverges} \\ 1 & \text{Test fails} \end{cases}$

The endpoints happen when $|x-a| = R$, Ratio test fails

Do NOT use the Ratio test on the endpoints — it won't work. *

$$X=-2: \quad \sum_{n=0}^{\infty} \left(\frac{-2}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$$

Divergence test $\lim_{n\to\infty} (-1)^n \begin{cases} 1 \\ -1 \end{cases} \neq 0$

diverges

$$X=2: \quad \sum_{n=0}^{\infty} \left(\frac{2}{2}\right)^n = \sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1 \qquad \text{Divergence test}$$

$\lim_{n\to\infty} 1 = 1 \neq 0$ diverges

Interval of Convergence: $(-2, 2)$

diverges at endpoints so they aren't included.

$\boxed{\text{Ex2}}$ $\quad \sum_{n=0}^{\infty} (2n)! \left(\frac{x}{2}\right)^n$

Ratio Test $\lim_{n\to\infty} \left| \frac{(2(n+1))! \cdot \left(\frac{x}{2}\right)^{n+1}}{(2n)! \left(\frac{x}{2}\right)^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)! \left(\frac{x}{2}\right)^{n+1}}{(2n)! \left(\frac{x}{2}\right)^n} \right| = \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)\cancel{(2n)!} \left(\frac{x}{2}\right)^{n+1}}{\cancel{(2n)!} \left(\frac{x}{2}\right)^n} \right|$$

$$= \lim_{n\to\infty} \left| (2n+2)(2n+1)\left(\frac{x}{2}\right) \right|$$

* To converge we need this limit to be less than 1. If $x = -1/100$, we'll have

$$\lim_{n\to\infty} \left| (2n+2)(2n+1)\left(\frac{-1}{200}\right) \right| = \infty.$$

In fact almost every value of x will let this go to ∞, except 1: when $x=0$

$$\lim_{n\to\infty} \left| (2n+2)(2n+1)\left(\frac{0}{2}\right) \right| = \lim_{n\to\infty} |0| = 0 < 1$$

* The interval of convergence is all values of x so that the series converges,
In this case I of C: $\{0\}$ $R=0$

we can't move away from $x=0$ without the series diverging

$\boxed{EX\,3}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{\sqrt{n}\,4^n}$

Ratio Test $\displaystyle\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{4^{n+1}\sqrt{n+1}} \cdot \frac{4^n\sqrt{n}}{(x+2)^n} \right| = \lim_{n\to\infty} \left| \frac{(x+2)}{4}\frac{\sqrt{n}}{\sqrt{n+1}} \right| = \left| \frac{x+2}{4} \right| < 1$

$|x+2| < 4$ $\quad R=4 \quad a=-2$

<u>End points:</u> $x = a+R \quad x = a-R$

$x = 2 \quad x = -6$

$$x=-6: \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-6+2)^n}{\sqrt{n}\,4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-4)^n}{\sqrt{n}\,4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{\sqrt{n}} = \sum_{n=1}^{\infty} -\frac{1}{\sqrt{n}}$$

* Note: This only looks like its Alternating — it is always negative. Check the 1st couple of terms

P4

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$ p-series

$p = 1/2 < 1$ diverges

$$x = 2: \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2+2)^n}{\sqrt{n}\ 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cancel{4^n}}{\sqrt{n}\ \cancel{4^n}}$$

Alternating Series Test

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

$C_1 = \frac{1}{\sqrt{1}} = 1 \quad C_2 = \frac{1}{\sqrt{2}} \quad C_3 = \frac{1}{\sqrt{3}}$

$1 > 1/\sqrt{2} > 1/\sqrt{3}$ ✓

converges

Interval of convergence

$(-6, 2]$

diverges at $x=-6$

converges at $x=2$

$\boxed{Ex\ 4}$ a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

Ratio test $\lim_{n\to\infty} \left| \frac{(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(x-1)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(x-1)(n+1)}{(n+2)} \right|$

$= |x-1| < 1$

$a=1$ $\quad R=1$

$|x-a| < R$

P5

Endpoints $x=a-R$ $\quad$ $x=a+R$

$x=1-1=0$ $\quad$ $x=1+1=2$

$x=0$: $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n+2}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$

$= \sum_{n=0}^{\infty} \frac{1}{n+1}$

* $n=0$ $(-1)^{0+2}=1$
  $n=1$ $(-1)^{2+2}=1$
  $n=2$ $(-1)^{4+2}=1$
  etc Not alternating

$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic series diverges

* If you hadn't realized that this was the Harmonic series, Limit comparison would have also worked.

$x=2$: $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$

$= -1 + \frac{1}{2} - \frac{1}{3} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Alternating Harmonic series converg

* Alternating series test would also work.

Interval of convergence $(0,2]$

P6

b) Find the derivative of the series from part a) & find its radius & interval of convergence

$$f'(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\cancel{(n+1)} (x-1)^n}{\cancel{n+1}}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} (x-1)^n$$

$R = 1$ (Radius doesn't change when we take derivatives / integrate)

End points $x=0$, $x=2$

$x=0$: $\sum_{n=0}^{\infty} (-1)^{n+1} (-1)^n = \sum_{n=0}^{\infty} (-1)^{2n+1} = \sum_{n=0}^{\infty} -1$

Divergence test $\lim_{n\to\infty} -1 = -1 \neq 0$
diverges

$x=2$: $\sum_{n=0}^{\infty} (-1)^{n+1} (2-1)^n = \sum_{n=0}^{\infty} (-1)^{n+1}$

Divergence test $\lim_{n\to\infty} (-1)^{n+1} \begin{cases} -1 \\ 1 \end{cases} \neq 0$
diverges

Interval of convergence
$(0, 2)$

Note: I of C doesn't necessarily stay the same