Stokes' Theorem

Let S be an oriented smooth surface that is bounded by a simple closed smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field then,

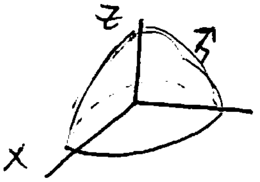
$$\int_C \vec{F}\cdot d\vec{r} = \iint_C \text{curl}\,\vec{F}\cdot d\vec{S}$$

Examples from Calculus 6e Matrix Version By Edwards & Perney

Use Stokes' theorem to evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

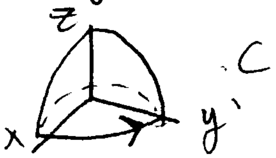
$\boxed{\text{Ex 1}}$

$\vec{F} = 3y\hat{\imath} - 2x\hat{\jmath} + xyz\hat{k}$; S is the hemispherical surface $z = \sqrt{4-x^2-y^2}$ with upper unit normal vector



* The directions want us to calculate $\int_C \vec{F}\cdot d\vec{r}$

First we'll identify C the boundary curve. In this case it will be $0=\sqrt{4-x^2y^2}$ or $x^2+y^2=4$ with $z=0$. Because we have upward unit normal the right hand rule gives us counter clockwise orientation



the curve C is described by

$\vec{r} = \langle 2\cos t, 2\sin t, 0\rangle$

remember we want only the edge of the circle.

P2

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle 3(2\sin t), -2(2\cos t), (2\cos t)(2\sin t)(0)\rangle \cdot \vec{r}\,'(t)\,dt$$

$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle$$

$$= \int_0^{2\pi} -12\sin^2 t - 8\cos^2 t \; dt$$

$$= \int_0^{2\pi} -8(\underbrace{\sin^2 t + \cos^2 t}_{1}) - 4\sin^2 t \; dt$$

$$= \int_0^{2\pi} -8 - \frac{4}{2}(1-\cos 2t)\,dt$$

$$= -8t - 2t + \frac{\cancel{2}}{\cancel{2}}\sin 2t \Big|_0^{2\pi}$$

$$= \boxed{-20\pi}$$

* Let's see if that was faster than calculating

$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & xyz \end{vmatrix}$$

$$= (xz - 0)\hat{\imath} - (yz - 0)\hat{\jmath} + (-2-3)\hat{k}$$

$$= \langle xz, -yz, -5\rangle$$

$z = \sqrt{4-x^2-y^2}$ $\quad \vec{n} = \vec{r}_x \times \vec{r}_y = -\frac{\partial z}{\partial x}\hat{\imath} - \frac{\partial z}{\partial y}\hat{\jmath} + \hat{k}$

$$= \frac{x}{\sqrt{4-x^2-y^2}}\hat{\imath} + \frac{y}{\sqrt{4-x^2-y^2}} + \hat{k}$$

$$\text{curl}\,\vec{F}\cdot\vec{n} = \frac{x^2 \cancel{z}}{\cancel{\sqrt{4-x^2-y^2}}} - \frac{y^2 \cancel{z}}{\cancel{\sqrt{4-x^2-y^2}}} + (-5)$$

$$= x^2 - y^2 - 5$$

$$\int_0^{2\pi}\int_0^2 (r^2\cos^2\theta - r^2\sin^2\theta - 5)\, r\, dr\, d\theta$$

P3

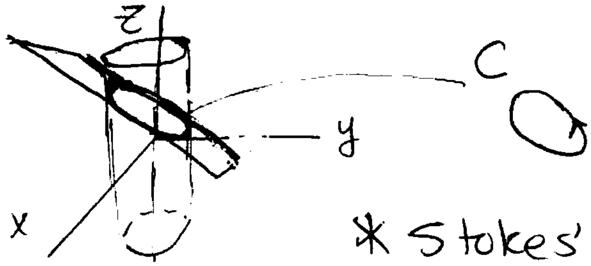
$$\int_0^{2\pi}\int_0^2 \left[r(\cos^2\theta - \sin^2\theta) - 5\right] r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 \left[r\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta - \left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right)\right) - 5\right] r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 r^2\cos 2\theta - 5r \, dr\,d\theta$$
$$= \int_0^{2\pi} \tfrac{8}{3}\cos 2\theta - \tfrac{5}{2}(4)\, d\theta$$
$$= +\tfrac{4}{3}\sin 2\theta \Big|_0^{2\pi} - 10\theta\Big|_0^{2\pi} = -20\pi$$

Easier the other way in this case

Use Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$

$\boxed{\text{Ex 2}}$

$\vec{F} = 2z\hat{i} + x\hat{j} + 3y\hat{k}$; C is the ellipse in which the plane $z = x$ meets the cylinder $x^2+y^2=4$ oriented counterclockwise as viewed from above



* Stokes' thm tells us $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & x & 3y \end{vmatrix} = \langle 3, -(-2), 1\rangle = \langle 3, 2, 1\rangle$$

We can have our surface S be any smooth one w/ boundary C but let's make our lives easier by making it the solid ellipse of the intersection

The right hand rule gives us upward orientation of $\vec{n}$, the normal vector to S
Any vector normal to **S** will be normal to $z=x$, $\vec{n} = -\frac{\partial z}{\partial x}\hat{\imath} - \frac{\partial z}{\partial y}\hat{\jmath} + \hat{k}$
$= \langle -1, 0, 1 \rangle$

$\text{Curl}\,\vec{F} \cdot \vec{n} = -3 + 0 + 1 = -2$

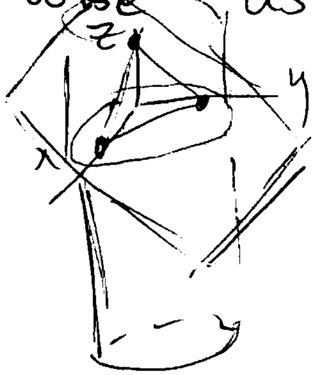
$\iint_D -2\, dA$ D is the projection of our surface on the xy plane which is $x^2+y^2=4$

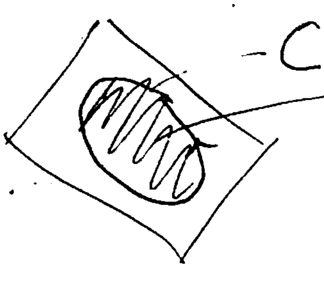
$\int_0^{2\pi}\int_0^2 -2\, r\, dr\, d\theta = \int_0^{2\pi} -4\, d\theta = \boxed{-8\pi}$

Examples from Stewart's Calculus Evaluate $\int_C \vec{F}\cdot d\vec{r}$

$\boxed{\text{EX3}}$

$\vec{F} = x^2 z\,\hat{\imath} + xy^2\,\hat{\jmath} + z^2\,\hat{k}$ and C is the curve of intersection of the plane $x+y+z=1$ and the Cylinder $x^2+y^2=9$ oriented counterclockwise as viewed from above.





We'll let S be the plane inside C again.

P5

$$\text{Curl}\,\vec F = \begin{vmatrix} \hat i & \hat j & \hat k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & xy^2 & z^2 \end{vmatrix} = \langle 0, -(0-x^2), y^2 \rangle = \langle 0, x^2, y^2 \rangle$$

Like the last problem, our normal vector will be normal to the plane $x+y+z=1$.

* Solve for z to use our shortcut

$z = 1-x-y \qquad \vec n = -\frac{\partial z}{\partial x}\hat i - \frac{\partial z}{\partial y}\hat j + \hat k = \langle 1,1,1 \rangle$

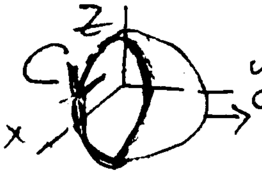
$\text{Curl}\,\vec F \cdot \vec n = x^2+y^2$

* If we project onto the xy plane we'll get the solid circle $x^2+y^2=9$ so using polar coordinates

$$\int_0^{2\pi}\int_0^3 (r^2)\,r\,dr\,d\theta = \frac{3^4}{4}2\pi = \boxed{\frac{81\pi}{2}}$$

$\boxed{\text{Ex 4}}$ Verify that Stokes' Thm is true for the given vector field $\vec F$ & surface S.

$\vec F = y\hat i + z\hat j + x\hat k$ $\quad S$ is the hemisphere $x^2+y^2+z^2=1$, $\boxed{y \geq 0}$ oriented in the direction of the positive y-axis



$\int_C \vec F \cdot d\vec r$

C is the circle $x^2+0^2+z^2=1$

so $\vec r(t) = \langle \cos t, 0, \sin t \rangle \qquad \vec r\,'(t) = \langle \sin t, 0, \cos t \rangle$

$\vec F(\vec r(t)) = \langle 0, \sin t, -\cos t \rangle$

$$\int_C \vec F \cdot d\vec r = \int_0^{2\pi} \langle 0, \sin t, -\cos t \rangle \cdot \langle \sin t, 0, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} -\cos^2 t\, dt = \int_0^{2\pi} -\frac{1}{2}(1+\cos 2t)\,dt = -\frac{1}{2}t - \frac{1}{4}\sin 2t \Big|_0^{2\pi}$$

* Note $\vec r$ is a little tricky because of our direction $\quad = \boxed{-\pi}$

P6

* Now we need to check that

$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = -\pi$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \langle -1, -1, -1 \rangle$$

$\vec{n}$ is normal to xz plane in the direction of positive $\hat{j}$ $\vec{n} = \langle 0, 1, 0 \rangle$

$\operatorname{curl} \vec{F} \cdot \langle 0, 1, 0 \rangle = -1$

$$\int_0^{2\pi} \int_0^1 -r \, dr \, d\theta = \int_0^{2\pi} -\frac{1}{2} \, d\theta = \boxed{-\pi} \checkmark$$