

# Surface Area

\* Area of a surface  $S$  with position vector  $\vec{r}(u,v)$  is

$$\iint_D |\vec{r}_u \times \vec{r}_v| du dv$$

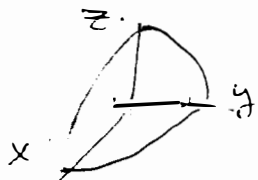
\* If  $S$  is given by  $z = f(x,y)$  then the area of  $S$  is  $\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$

Examples from Calculus: Early Transcendental Functions By Larson, Hostetler, Edwards

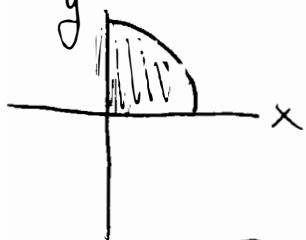
**Ex 1** Find the area of the portion of the paraboloid  $z = 16 - x^2 - y^2$  in the 1<sup>st</sup> octant

\* Our surface is  $z = 16 - x^2 - y^2$  so we can use our shortcut

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + (-2x)^2 + (-2y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$



$z$  depends on  $x$  and  $y$  so our domain is given by what is going on in the  $xy$  plane. Imagine taking the portion of our surface above the  $xy$  plane & also in the 1<sup>st</sup> octant and smooching it on to the  $xy$ -plane. This would give us



$xy$ -plane  $\rightarrow z = 0$  so  $16 = x^2 + y^2 = r^2$   
Using polar coordinates we get

$$\text{Surface area} = \int_0^{\frac{\pi}{2}} \int_0^4 \sqrt{1 + 4r^2} r dr d\theta$$

Note! We could have described  $Z = 16 - x^2 - y^2$  with polar coordinates from the start.

$$Z = 16 - r^2$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\vec{r}_\theta = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

$$\vec{r}_r = \langle \cos \theta, \sin \theta, -2r \rangle$$

$$\vec{r}_\theta \times \vec{r}_r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & -2r \end{vmatrix} = \langle -2r^2 \cos \theta, -2r^2 \sin \theta, -r \sin^2 \theta - r \cos^2 \theta \rangle$$

$$= \langle -2r^2 \cos \theta, -2r^2 \sin \theta, -r \rangle$$

$$|\vec{r}_\theta \times \vec{r}_r| = \sqrt{(-2r^2 \cos \theta)^2 + (-2r^2 \sin \theta)^2 + (-r)^2}$$

$$= \sqrt{4r^4 + r^2}$$

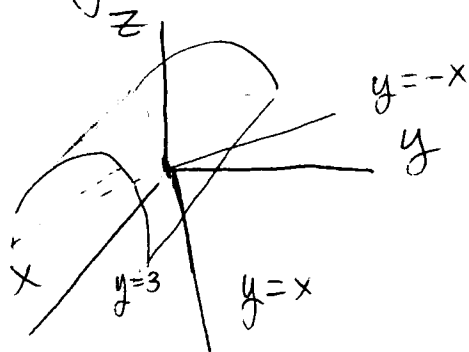
$$= r \sqrt{4r^2 + 1}$$

$$A(S) = \iint_D |\vec{r}_\theta \times \vec{r}_r| \, dr \, d\theta = \int_0^{\frac{\pi}{2}} \int_0^4 \sqrt{4r^2 + 1} \, r \, dr \, d\theta$$

\* We didn't need to put  $r \, dr \, d\theta$  since by the definition we just need  $|\vec{r}_\theta \times \vec{r}_r| \, dr \, d\theta$ . It is when we convert to polar coordinates after the cross product we need  $r \, dr \, d\theta$ , since  $dy \, dx = r \, dr \, d\theta$ . Obviously the 1st method was much easier. \*

P3

**Ex2** Find the area of the surface of the cylinder  $f(x,y) = 9-y^2$  that lies above the triangle bounded by  $y=x$ ,  $y=-x$  &  $y=3$

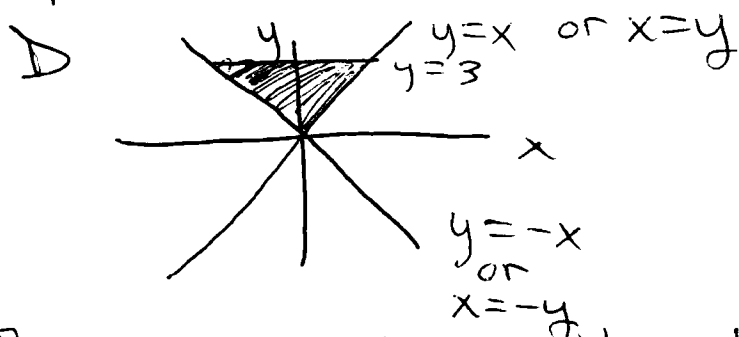


Surface is  $z = 9 - y^2$   
we can use the shortcut

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + (0)^2 + (-2y)^2}$$

$$= \sqrt{1 + 4y^2}$$

Now we need to find



This is most easily described as a type II region

$$\int_0^3 \int_{-y}^y \sqrt{1+4y^2} dx dy = \int_0^3 2y \sqrt{1+4y^2} dy$$

$$u = 1 + 4y^2$$

$$du = 8y dy$$

$$\frac{1}{8} du = y dy$$

$$= \frac{1}{8} \int_1^{37} 2 u^{1/2} du$$

$$= \frac{1}{4} \left[ \frac{2}{3} u^{3/2} \right]_1^{37}$$

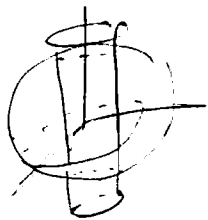
$$u(0) = 1 + 4(0^2) = 1$$

$$u(3) = 1 + 4(3^2) = 37$$

$$= \frac{1}{4} \left[ \frac{2}{3} (37^{3/2} - 1) \right]$$

$$= \frac{1}{6} [37^{3/2} - 1]$$

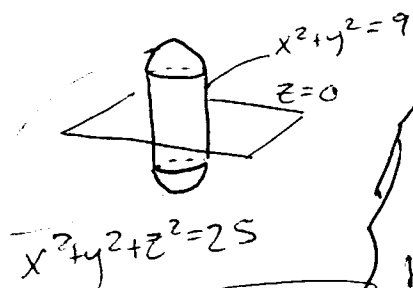
P4 **EX3** Find the area of the portion of the sphere  $x^2 + y^2 + z^2 = 25$  inside the cylinder  $x^2 + y^2 = 9$



$$\text{Surface} \Rightarrow x^2 + y^2 + z^2 = 25$$

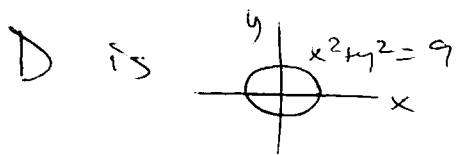
Solve for  $z$  & use the shortcut

$$z = \sqrt{25 - x^2 - y^2}$$



Note! This only gives us the surface above  $z=0$ . We will need to multiply our result by 2 to get the area of the whole region

$$\begin{aligned} |\vec{r}_x \times \vec{r}_y| &= \sqrt{1 + \left(\frac{-x}{\sqrt{25-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{25-x^2-y^2}}\right)^2} \\ &= \sqrt{\frac{25-x^2-y^2}{25-x^2-y^2} + \frac{x^2}{25-x^2-y^2} + \frac{y^2}{25-x^2-y^2}} \\ &= \frac{5}{\sqrt{25-x^2-y^2}} \end{aligned}$$



again polar coordinates will make this problem easy (relatively)

$$SA = 2 \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25-r^2}} r dr d\theta$$

$$u = 25 - r^2 \quad u(0) = 25$$

$$du = -2r dr \quad u(3) = 16$$

$$\frac{1}{2} du = -r dr$$

P5

$$\begin{aligned}
 S(A) &= 2 \int_0^{2\pi} \int_{25}^{16} \frac{5}{\sqrt{u}} du d\theta \\
 &= \int_0^{2\pi} \int_{16}^{25} 5u^{-1/2} du d\theta \\
 &= 2\pi \left[ 10\sqrt{u} \right]_{16}^{25} = 20\pi [5-4] = \boxed{20\pi}
 \end{aligned}$$

**EX4** From Stewart's Calculus p900

\* 38. Set up an integral for the area of the parametric surface given by the vector function

$$\vec{r}(u,v) = v^2\hat{i} - uv\hat{j} + u^2\hat{k} \quad 0 \leq u \leq 3$$

$$-3 \leq v \leq 3$$

\* Sadly our shortcut won't work here

$$\vec{r}_u = \langle 0, -v, 2u \rangle \quad \vec{r}_v = \langle 2v, -u, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -v & 2u \\ 2v & -u & 0 \end{vmatrix} = \langle 2u^2, -4uv, 2v^2 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4u^4 + 16u^2v^2 + 4v^4}$$

↑ If this were an 8 we could factor this - alas.

$$S(A) = \int_0^3 \int_{-3}^3 \sqrt{4u^4 + 16u^2v^2 + 4v^4} dv du$$