Taylor Polynomials

Taylor polynomials are just finite Taylor series. They are useful when we want to approximate functions

$T_n(x) = n^{th}$ degree Taylor polynomial

$$= f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Technique: ① Find all derivatives of $f(x)$ up to $f^{(n)}(x)$

② Evaluate f & its derivatives at a

③ Plug into

$$f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$\boxed{\text{Ex 1}}$ Find $T_2(x)$ for $f(x) = \sec x$ at $a = 0$

$\to f(x) = \sec x$

$f'(x) = \sec x \tan x$

$f''(x) = (\sec x \tan x)\tan x + \sec x(\sec^2 x)$ Product rule!

we just need up to the 2^{nd} derivative since they want T_2

→ Plug in a

$f(0) = \sec 0 = \frac{1}{\cos 0} = 1$

$f'(0) = \sec 0 \tan 0 = 1 \cdot 0 = 0$

$f''(0) = (\sec 0 \tan 0) \tan 0 + \sec^3 0 = 0 + 1 = 1$

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$$

$$T_2(x) = 1 + 0(x-0) + \frac{1(x-0)^2}{2!}$$
$$= 1 + \frac{x^2}{2!}$$
$$= 1 + \frac{x^2}{2}$$

$\sec x \approx 1 + \frac{x^2}{2}$ when x is near 0

easier to use than $\sec x$

$\boxed{\text{Ex 2}}$ a) Find $T_4(x)$ for $f(x) = \frac{1}{\sqrt{x}}$ $a = 1$

→ We need to find up to $f^{(4)}(x)$

$f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$f'(x) = -\frac{1}{2}x^{-3/2}$

$f''(x) = \frac{3}{4}x^{-5/2}$

$f'''(x) = -\frac{15}{8}x^{-7/2}$

$f^4(x) = \frac{15(7)}{16}x^{-9/2}$

$\rightarrow f(1) = 1$

$f'(1) = -\frac{1}{2}$

$f''(1) = \frac{3}{4}$

$f'''(1) = -\frac{15}{8}$

$f^4(1) = \frac{15(7)}{16}$

$\rightarrow T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \frac{f^4(a)(x-a)^4}{4!}$

$$T_4(x) = 1 - \frac{1}{2}(x-1) + \frac{\frac{3}{4}(x-1)^2}{2!} + \frac{\left(-\frac{15}{8}\right)(x-1)^3}{3!} + \frac{\frac{15(7)}{16}(x-1)^4}{4!}$$

b) Approximate $\frac{1}{\sqrt{2}}$ with $T_4(x)$

We know $\frac{1}{\sqrt{x}} \approx T_4(x)$ so

$$\frac{1}{\sqrt{2}} \approx T_4(2) = 1 - \frac{1}{2}(2-1) + \frac{\frac{3}{4}(2-1)^2}{2!} - \frac{\frac{15}{8}(2-1)^3}{3!} + \frac{\frac{15(7)}{16}(2-1)^4}{4!}$$

$$= 1 - \frac{1}{2} + \frac{3}{4}\frac{1}{2!} - \frac{15}{8 \cdot 3!} + \frac{15(7)}{16 \cdot 4!}$$

We could have gotten a more accurate approximation by using a higher degree Taylor polynomial

$\boxed{\text{Ex 3}}$ Find $T_3(x)$ of $f(x) = e^x$, $a = -2$

$\rightarrow f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

$\rightarrow f(-2) = e^{-2}$

$f'(-2) = e^{-2}$

$f''(-2) = e^{-2}$

$f'''(-2) = e^{-2}$

want T_3, take up to the 3rd derivative

$$\rightarrow T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$$

$$T_3 = e^{-2} + e^{-2}(x+2) + \frac{e^{-2}(x+2)^2}{2!} + \frac{e^{-2}(x+2)^3}{3!}$$