P1

Undetermined Coefficients w/ Matrices

The technique of Undetermined Coefficients doesn't change much from when we first learned it dealing with $ay''+by'+cy=f(t)$. However dealing with overlap between the complementary and particular solutions can get tricky. We'll just focus on the cases where there is no overlap since Variation of parameters can deal with the other cases.

The system we're dealing with is $\vec{X}'=A\vec{X}+\vec{f}$ where $\vec{f}\neq\vec{0}$

P2 Examples from Differential Equations by Edwards & Penney

$\boxed{\text{Ex1}}$ $\vec{x}\,' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

① If we wanted to find the general solution, we would find the eigenvalues & eigenvectors of A, form the complementary solution & then add that to the particular solution. But I don't want to do that :)

- 1st we find the form of $\vec{x}_p$.
  Look at $\vec{f}$ if it contains a polynomial match the degree & decrease until you reach a constant vector.
- In this case $\vec{x}_p = \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$
- once you have $\vec{x}_p$, find $\vec{x}_p\,'$
  $\vec{x}_p\,' = \vec{0}$ (YAY!)
- Plug into $\vec{x}\,' = A\vec{x} + \vec{f}$

$$\vec{0} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

The first line says $0 = a_1 + 2a_2 + 3$

& the second line says $0 = 2a_1 + a_2 - 2$

or $a_1 + 2a_2 = -3 \rightarrow -2a_1 - 4a_2 = 6$

$2a_1 + a_2 = 2$

$$\begin{array}{r} -2a_1 - 4a_2 = 6 \\ 2a_1 + a_2 = 2 \\ \hline -3a_2 = 8 \end{array}$$

$a_2 = \frac{-8}{3}$

$a_1 = -3 - 2\left(\frac{-8}{3}\right) = -\frac{9}{3} + \frac{16}{3} = \frac{7}{3}$

$$\boxed{\vec{X_p} = \begin{pmatrix} 7/3 \\ -8/3 \end{pmatrix}}$$

Ex2 $\vec{X}' = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix} \vec{X} + \begin{pmatrix} 0 \\ t^2 \end{pmatrix}$

- Form of $\vec{X_p} = \vec{a}t^2 + \vec{b}t + \vec{c}$
- $\vec{X_p}' = 2\vec{a}t + \vec{b}$
- Plug in

$$2\vec{a}t + \vec{b} = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}\vec{a}t^2 + \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}\vec{b}t + \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}\vec{c} + \begin{pmatrix} 0 \\ t^2 \end{pmatrix}$$

* This might not be fun

- Group like terms together

All the t^2's:

$$\vec{0}t^2 = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}\vec{a}t^2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}t^2$$

All the t's:

$$2\vec{a}t = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}\vec{b}t$$

All the constants:

$$\vec{b} = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}\vec{c}$$

- The first one only has $\vec{a}$'s as the unknown, so we'll solve it first.

$$\vec{0} = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} 0 &= 3a_1 + 4a_2 + 0 \\ 0 &= 3a_1 + 2a_2 + 1 \end{aligned}\right\} \quad \begin{aligned} 0 &= 2a_2 - 1 \\ a_2 &= \tfrac{1}{2} \end{aligned}$$

$$3a_1 = -4a_2$$

$$a_1 = -2/3$$

$$\vec{a} = \begin{pmatrix} -2/3 \\ 1/2 \end{pmatrix} \longrightarrow \text{Plug into } 2^{nd} \text{ equation}$$

p5

$$2\begin{pmatrix} -2/3 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$-4/3 = 3b_1 + 4b_2$$

$$1 = 3b_1 + 2b_2$$

$$-\frac{7}{3} = 2b_2 \quad b_2 = -\frac{7}{6}$$

$$1 = 3b_1 + 2\left(-\frac{7}{6}\right)$$

$$3b_1 = \frac{20}{6} = \frac{10}{3}$$

$$b_1 = \frac{10}{9} \qquad \vec{b} = \begin{pmatrix} 10/9 \\ -7/6 \end{pmatrix}$$

- Plug into the 3rd equation

$$\begin{pmatrix} 10/9 \\ -7/6 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\frac{20}{18} = 10/9 = 3c_1 + 4c_2$$

$$\frac{-21}{18} = -7/6 = 3c_1 + 2c_2$$

$$\frac{41}{18} = 2c_2 \qquad c_2 = \frac{41}{36}.$$

$$-\frac{21}{18} = 3c_1 + \frac{41}{18} \qquad c_1 = \frac{-62}{18(3)} = \frac{-31}{27}$$

$$\vec{X_p} = \begin{pmatrix} -2/3 \\ 1/2 \end{pmatrix}t^2 + \begin{pmatrix} 10/9 \\ -7/6 \end{pmatrix}t + \begin{pmatrix} -\frac{31}{27} \\ 41/36 \end{pmatrix}$$

p6 $\boxed{\text{Ex 3}}$ $\vec{x}' = \begin{pmatrix} 6 & -7 \\ 4 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 10 \\ -2e^{-t} \end{pmatrix}$

$\vec{X}_p = \vec{a} + \vec{b}e^{-t}$

for the 10

$\vec{X}_p' = -\vec{b}e^{-t}$

$-\vec{b}e^{-t} = \begin{pmatrix} 6 & -7 \\ 4 & -2 \end{pmatrix} \vec{a} + \begin{pmatrix} 6 & -7 \\ 4 & -2 \end{pmatrix} \vec{b}e^{-t} + \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^{-t}$

Group like terms

$-\vec{b} = \begin{pmatrix} 6 & -7 \\ 4 & -2 \end{pmatrix} \vec{b} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \longrightarrow \begin{matrix} -b_1 = 6b_1 - 7b_2 + 0 \\ -b_2 = 4b_1 - 2b_2 - 2 \end{matrix}$

$\vec{0} = \begin{pmatrix} 6 & -7 \\ 4 & -2 \end{pmatrix} \vec{a} + \begin{pmatrix} 10 \\ 0 \end{pmatrix} \longrightarrow \begin{matrix} 0 = 6a_1 - 7a_2 + 10 \\ 0 = 4a_1 - 2a_2 + 0 \\ 0 = 2a_1 - a_2 \end{matrix}$

$0 = 7b_1 - 7b_2 \quad \leftarrow \quad 0 = b_1 - b_2$

$2 = 4b_1 - b_2 \qquad 2 = 4b_1 - b_2$

$2 = 3b_1 \quad b_1 = 2/3$

$b_2 = 2/3$

$0 = -8a_1 + 10$

$a_1 = \frac{10}{8} = \frac{5}{4}$

$a_2 = 5/2$

$$\boxed{\vec{X}_p = \begin{pmatrix} 5/4 \\ 5/2 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} e^{-t}}$$