7.6 Unit Step Functions

Def: $u(t-a) = \begin{cases} 0 & t<a \\ 1 & a\le t \end{cases}$ Alternative def

Ex from Introduction to Differential equations
By Campbell & Haberman

Write the function defined on $[0, \infty)$
in terms of unit step functions

EX 1 $f(t) = \begin{cases} 0 & 0 \le t < 2 \\ 3 & 2 \le t < 5 \\ t & 5 \le t \end{cases}$

* Think of unit step functions like a switch. & when they are off they equal zero & when they're on they equal 1. But unlike a switch once they're on, they stay on. Because of this start the problem with the 1st value of f(t) and go from there *

$f(t) = 0 \quad 0 \le t < 2$

Any unit step function with $a \ge 2$ will do the job. Let's look at the next requirement.

$f(t) = 3 \quad \boxed{2 \le t} < 5$

This tells us $u(t-2)$ is involved.

$f(t) = 3u(t-2)$

Check that this satisfies $f(t) = \begin{cases} 0 & 0 \le t < 2 \\ 3 & 2 \le t < 5 \end{cases}$

If $t < 2$ $u(t-2) = 0$ ✓

If $t \ge 2$ $u(t-2) = 1$

so $f(t) = 3u(t-2)$ works so far.

Now we need $f(t) = t$ if $\boxed{5 \le t}$

This tells us $u(t-5)$ is involved.

$$f(t) = 3u(t-2) + \underline{\quad\quad} u(t-5)$$

This function is still "on"

so $f(t) = 3 + \underline{t-3} = t$ if $t \ge 5$

$$\boxed{f(t) = 3u(t-2) + (t-3)u(t-5)}$$

Ex 2

$$f(t) = \begin{cases} 1 & \boxed{0 \le t} < 1 \\ -1 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ -1 & 3 \le t \end{cases}$$

We can use $u(t)$ or just 1 here

$f(t) = u(t)$ satifies $f(t) = 1$ $0 \le t < 1$

Now we need $f(t) = -1$ $\boxed{1 \le t} < 2$

$u(t-1)$ is

$$f(t) = u(t) + \underline{\qquad}\, u(t-1) = -1 \text{ if } 1 \le t < 2$$
$$= 1 + \underline{\qquad} = -1 \text{ if } 1 \le t < 2$$

so $f(t) = u(t) - 2u(t-1)$ works for $t < 2$

Next we need $f(t) = 1$ $\boxed{2 \le t} < 3$

$u(t-2)$

$$f(t) = u(t) - 2u(t-1) + \underline{\qquad}\, u(t-2) = 1$$
if $2 \le t < 3$

If $2 \le t < 3$ $f(t) = 1 - 2 + \underline{\qquad} = 1$

so $f(t) = u(t) - 2u(t-1) + 2u(t-2)$ for $t < 3$

Lastly $f(t) = -1$ if $\boxed{3 \le t}$

$$f(t) = u(t) - 2u(t-1) + 2u(t-2) + \underline{\qquad}\, u(t-3)$$
$$= 1 - 2 + 2 + \underline{\qquad} = -1 \text{ if } t \ge 3$$

so $\boxed{f(t) = u(t) - 2u(t-1) + 2u(t-2) - 2u(t-3)}$

$$f(t)=\begin{cases}\sin t & 0\le t<\pi\\ 0 & \pi\le t<2\pi\\ \sin t & 2\pi\le t<3\pi\\ 0 & 3\pi\le t\end{cases}$$

$f(t)=\sin t$ or $f(t)=\sin t\, u(t)$ · both work for $0\le t<\pi$

$f(t)=0$ if $\boxed{\pi\le t}<2\pi$

$f(t)=\sin t + \underline{\qquad} u(t-\pi)=0$

so $f(t)=\sin t-\sin t\,u(t-\pi)$

$f(t)=\sin t$ if $\boxed{2\pi\le t}<3\pi$

$f(t)=\sin t-\sin t\,u(t-\pi)+\sin t\,u(t-2\pi)$

$f(t)=0 \quad 3\pi\le t$

$$\boxed{f(t)=\sin t-\sin t\,u(t-\pi)+\sin t\,u(t-2\pi)-\sin t\,u(t-}$$

From Differential Equations & BVP
By Nagle, Saff, & Snider

P397 Solve the IVP using the method of Laplace transforms

31. $y''+4y=g(t) \quad y(0)=1 \quad y'(0)=3$

Where $g(t)=\begin{cases} \sin t & 0\le t\le 2\pi \\ 0 & 2\pi<t \end{cases}$

so $g(t)=\sin t-\sin t\, u(t-2\pi)$

$$y''+4y=\sin t-\sin t\, u(t-2\pi)$$

$$\mathcal{L}\{y''\}+4\mathcal{L}\{y\}=\mathcal{L}\{\sin t\}-\mathcal{L}\{\sin t\, u(t-2\pi)\}$$

$$s^2\mathcal{L}\{y\}-sy(0)-y'(0)+4\mathcal{L}\{y\}=\frac{1}{s^2+1}-e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\}$$

$$s^2\mathcal{L}\{y\}-s-3+4\mathcal{L}\{y\}=\frac{1}{s^2+1}-e^{-2\pi s}\mathcal{L}\{\sin t\}$$ TRICKY

$$s^2\mathcal{L}\{y\}-s-3+4\mathcal{L}\{y\}=\frac{1}{s^2+1}-e^{-2\pi s}\frac{1}{s^2+1}$$

$$\mathcal{L}\{y\}=\frac{\frac{1}{s^2+1}-e^{-2\pi s}\frac{1}{s^2+1}+s+3}{s^2+4}$$

$$=\frac{1}{(s^2+1)(s^2+4)}-e^{-2\pi s}\frac{1}{(s^2+1)(s^2+4)}+\frac{s+3}{s^2+4}$$

$$\boxed{\frac{1}{(s^2+1)(s^2+4)}}$$ We need to make it easier to find $\mathcal{L}^{-1}$ of this

$$= \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$(As+B)(s^2+4) + (Cs+D)(s^2+1) = 1$$

$$As^3 + \boxed{4As} + Bs^2 + 4B + Cs^3 + \boxed{Cs} + Ds^2 + D = 1$$

$$A+C=0 \qquad A=-C$$

$$B+D=0 \qquad B=-D$$

$$4A+C=0 \qquad -3C=0 \quad C=0 \rightarrow A=0$$

$$4B+D=1$$

$$-4D+D=-3D=1 \quad D=-1/3 \quad B=1/3$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{\frac{1}{3}}{s^2+1} - \frac{\frac{1}{3}}{s^2+4}$$

$$\mathcal{L}\{y\} = \left(\frac{\frac{1}{3}}{s^2+1} - \frac{\frac{1}{3}}{s^2+4}\right) - e^{-2\pi s}\left(\frac{1/3}{s^2+1} - \frac{\frac{1}{3}}{s^2+4}\right) + \frac{s}{s^2+4} + \frac{3}{s^2+4}$$

$$y = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \mathcal{L}^{-1}\left\{e^{-2\pi s}\left(\frac{\frac{1}{3}}{s^2+1} - \frac{\frac{1}{3}}{s^2+4}\right)\right\} + \cos 2t + \frac{3}{2}\sin 2$$

<u>Recall:</u> $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$

Here $F(s) = \frac{\frac{1}{3}}{s^2+1} - \frac{\frac{1}{3}}{s^2+4} \rightarrow f(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$

$f(t-2\pi) = $

$$y = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - u(t-2\pi)\left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin(2t-\right.$$

$$+ \cos 2t + \frac{3}{2}\sin 2t$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - u(t-2\pi)\left[\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right]$$

$$+ \cos 2t + \frac{3}{2}\sin 2t$$

$$\boxed{y = \frac{1}{3}\sin t + \frac{8}{6}\sin 2t - u(t-2\pi)\left[\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right] + \cos 2t}$$