Variation of Parameters

Variation-of Parameters is a method we use to solve $ay''+by'+cy=f(t)$ when Undetermined Coefficients fails

If $y_c=c_1y_1+c_2y_2$ is a general solution to $ay''+by'+cy=0$, then our $y_p=v_1y_1+v_2y_2$ where v_1 and v_2 are functions of t which satisfy

$$\begin{cases} v_1'y_1+v_2'y_2=0 \\ v_1'y_1'+v_2'y_2'=\dfrac{f(t)}{a} \end{cases}$$

Examples from Elementary D.E & BVPs by Boyce and Diprima

Find the general solution

$\boxed{\text{Ex 1}}$ $\overset{a}{1}y''+y=\overset{f(t)}{\tan t}$

① Find y_c, $r^2+1=0$ $r^2=-1$

$r=\pm i$

$y_c=c_1\underset{y_1}{\cos t}+c_2\underset{y_2}{\sin t}$

② Find y_p

So, $y_c = v_1 \cos t + v_2 \sin t$

③ Plug y_1, y_2, y_1', y_2', etc into equation

$$v_1' \cos t + v_2' \sin t = 0$$

$$v_1'(-\sin t) + v_2' \cos t = \frac{\tan t}{1}$$

④ Solve the 1st equation for v_1'

$$v_1' = -v_2' \frac{\sin t}{\cos t}$$

⑤ Plug v_1' into the 2nd equation & solve for v_2'

$$\left(-v_2' \frac{\sin t}{\cos t}\right)(-\sin t) + v_2' \cos t = \tan t$$

$$v_2' \left[\frac{\sin^2 t}{\cos t} + \cos t\right] = \tan t$$

* Multiply both sides by $\cos t$

$$v_2' \left[\sin^2 t + \cos^2 t\right] = \tan t \cos t = \left(\frac{\sin t}{\cancel{\cos t}}\right) \cancel{\cos t}$$

$$v_2' = \sin t$$

$$v_2 = -\cos t$$ No need for "+ C"

$$V_1' = -V_2' \frac{\sin t}{\cos t}$$

$$V_1' = -\sin t\left(\frac{\sin t}{\cos t}\right)$$

$$V_1' = -\frac{\sin^2 t}{\cos t}$$

$$V_1 = \int -\frac{\sin^2 t}{\cos t}\,dt$$

☆ This is where the problem gets tricky. Up to here this was a reasonable problem

$$= \int -\frac{(1-\cos^2 t)}{\cos t}\,dt$$

$$= \int \frac{-1}{\cos t} + \frac{\cos^2 t}{\cos t}\,dt$$

$$= \int -\sec t + \cos t\,dt$$

$$= -\ln|\sec t + \tan t| + \sin t$$

$$\int \sec t\,dt$$

$$= \int \sec t\left(\frac{\sec t + \tan t}{\sec t + \tan t}\right)dt$$

$$= \int \frac{\sec^2 t + \sec t\tan t}{\sec t + \tan t}\,dt$$

$$= \int \frac{1}{u}\,du \quad u = \sec t + \tan t$$

$$= \ln|\sec t + \tan t|$$

$$y_p = (-\ln|\sec t + \tan t| + \sin t)\cos t - \cos t\sin t$$

General solution $y = y_c + y_p$

$$y = c_1\cos t + c_2\sin t + (-\ln|\sec t + \tan t| + \sin t)\cos t - \cos t\sin t$$

$$\boxed{y = c_1\cos t + c_2\sin t - \ln|\sec t + \tan t|\cos t}$$

$\boxed{\text{Ex 2}}$ $y'' + 4y' + 4y = t^{-2}e^{-2t}$

It is this negative which makes the problem require variation of parameters

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$y_c = \underbrace{c_1 e^{-2t}}_{y_1} + c_2 \underbrace{te^{-2t}}_{y_2}$

$y_p = v_1 e^{-2t} + v_2 t e^{-2t}$

$v_1' e^{-2t} + v_2' t e^{-2t} = 0$

$v_1'(-2e^{-2t}) + v_2'(e^{-2t} - 2te^{-2t}) = \dfrac{t^{-2}e^{-2t}}{1}$

* Solve 1st equation for v_1'

$v_1' = \dfrac{-v_2' t \cancel{e^{-2t}}}{\cancel{e^{-2t}}} = -v_2' t$

* Plug into 2nd

$\underline{(-v_2' t)(-2e^{-2t})} + v_2'(e^{-2t} \underline{- 2te^{-2t}}) = t^{-2}e^{-2t}$

$v_2' e^{-2t} = t^{-2}e^{-2t}$

$$V_2' = t^{-2}$$

$$V_2 = -\frac{1}{t}$$

$$V_1' = -V_2' t$$
$$= -(t^{-2})t$$
$$= -t^{-1}$$

$$V_1 = -\ln|t|$$

General solution $\boxed{y = c_1 e^{-2t} + c_2 t e^{-2t} - \ln|t| e^{-2t} + \underbrace{\left(\frac{-1}{t}\right) t e^{-2t}}_{-e^{-2t}}}$

or

$$y = c_1 e^{-2t} + c_2 t e^{-2t} - \ln|t| e^{-2t}$$

c_1 absorbs

$\boxed{\text{Ex 3}}$ $y'' + 4y = 3\csc 2t$

$$r^2 + 4 = 0$$
$$r = \pm 2i$$

$$y_c = c_1 \cos 2t + c_2 \sin 2t$$

$$y_p = V_1 \cos 2t + V_2 \sin 2t$$

$$v_1' \cos 2t + v_2' \sin 2t = 0$$

$$v_1'(-2\sin 2t) + v_2'(2\cos 2t) = 3\csc 2t$$

$$v_1' = -v_2' \frac{\sin 2t}{\cos 2t}$$

(this will be similar to our 1st ex, but <u>easier</u> :))

$$\left(-v_2' \frac{\sin 2t}{\cos 2t}\right)(-2\sin 2t) + v_2'(2\cos 2t) = 3\csc 2t$$

$$v_2' 2\left(\frac{\sin^2 2t}{\cos 2t} + \cos 2t\right) = 3\csc 2t$$

$$v_2' 2(\sin^2 2t + \cos^2 2t) = 3\csc 2t \cos 2t$$

$$v_2' = \frac{3}{2}\frac{\cos 2t}{\sin 2t}$$

$$v_2 = \int \frac{3}{2}\frac{\cos 2t}{\sin 2t}\,dt \qquad \begin{aligned} u &= \sin 2t \\ du &= 2\cos 2t\,dt \\ \tfrac{1}{2}du &= \cos 2t \end{aligned}$$

$$= \int \frac{3}{4}\frac{1}{u}\,du = \frac{3}{4}\ln|\sin 2t|$$

$$v_1' = \left(-\frac{3}{2}\frac{\cancel{\cos 2t}}{\cancel{\sin 2t}}\right)\left(\frac{\cancel{\sin 2t}}{\cancel{\cos 2t}}\right) \qquad \text{Yay!}$$

$$v_1' = -\frac{3}{2}$$

$$v_1 = -\frac{3}{2}t$$

General solution: $\boxed{y = c_1\cos 2t + c_2\sin 2t - \frac{3}{2}t\cos 2t + \frac{3}{4}\ln|\sin 2t|\sin 2t}$

$\boxed{\text{Ex 4}}$

$$y'' - 2y' + y = \frac{e^t}{(1+t^2)}$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$y_c = c_1 e^t + c_2 t e^t$$

$$y_p = v_1 e^t + v_2 t e^t$$

$$v_1' e^t + v_2' t e^t = 0$$

$$v_1' e^t + v_2'(e^t + te^t) = \frac{e^t}{(1+t^2)}$$

$$v_1' = \frac{-v_2' t \cancel{e^t}}{\cancel{e^t}}$$

$$(-v_2' t)e^t + v_2'(e^t + te^t) = \frac{e^t}{1+t^2}$$

$$v_2' e^t = \frac{e^t}{1+t^2}$$

$$v_2' = \frac{1}{1+t^2}$$

$$v_2 = \int \frac{1}{1+t^2}\,dt$$

$$= \tan^{-1} t$$

$$\int v_1' = \int -\frac{t}{1+t^2}\,dt \quad u = 1+t^2 \quad du = 2t\,dt \quad \frac{1}{2}du = t\,dt$$

$$v_1 = \int -\frac{1}{2}\frac{1}{u}\,du$$

$$= -\frac{1}{2}\ln(1+t^2)$$

$$\boxed{y = c_1 e^t + c_2 t e^t - \frac{1}{2}\ln(1+t^2)\,e^t + \tan^{-1}t\,(t\,e^t)}$$