U-Substitution

The point of u-substitution is to undo the chain rule.

Recall: $\underbrace{\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x)}_{\text{Chain rule}}$

We want to be able to integrate problems of the form:

$$\int f'(g(x))\,g'(x)\,dx$$

Let $u = g(x)$, the "inside function"

$du = g'(x)dx$

so our integral becomes

$$\int f'(u)\,du = f(u) + C = f(g(x)) + C$$

$\boxed{\text{Ex 1}}$ $\int 3\cos(3x)\,dx$

☆ we need to look for the inside function, the $g(x)$.

we can see the $3x$ inside the cosine function

$$u = 3x$$
$$du = 3\,dx$$

$$\int 3\cos(3x)\,dx = \int \cos(3x)\underbrace{3\,dx}_{du}$$

$$= \int \cos u\,du = \sin u + C$$

$$= \boxed{\sin(3x) + C}$$

☆ This problem had du perfectly matched, but that doesn't always happen

$\boxed{\text{Ex 2}}$ $\int \cos(16x)\,dx$

★ Again we're looking for the "inside function"

$u = 16x$

$du = 16\,dx$, we have the dx but are missing the 16. This is fine. We are are just off by a constant which will happen a lot.

$du = 16\,dx$

$\frac{1}{16}du = dx$

$$\int \cos(16x)\underbrace{dx}_{\frac{1}{16}du} = \int \cos(u)\underbrace{\tfrac{1}{16}du}_{dx}$$

$$= \frac{1}{16}\int \cos u\,du = \frac{1}{16}\sin u + C = \boxed{\frac{1}{16}\sin(16x) + C}$$

Integrals of the forms

$\int \cos(ax)\,dx$, $\int \sin(ax)\,dx$, $\int e^{ax}\,dx$ happen so often that I strongly recommend students who are in Calculus 2 or higher to memorize the results.

$\boxed{\text{Ex 3}}$ $\int \cos(ax)\,dx$

$u = ax$

$du = a\,dx$

$\frac{1}{a}du = dx$

$$\int \cos(ax)\,dx = \int \cos u \underbrace{\tfrac{1}{a}du}_{dx}$$

$$= \frac{1}{a}\int \cos u\,du = \frac{1}{a}\sin u + C = \boxed{\frac{1}{a}\sin ax + C}$$

Ex 4 $\int \sin(ax)\,dx$

$u = ax$

$du = a\,dx$

$\frac{1}{a}du = dx$

$$\int \sin(ax)\,dx = \int \frac{1}{a}\sin u\,du$$

$$= -\frac{1}{a}\cos u + C = \boxed{-\frac{1}{a}\cos(ax) + C}$$

Ex 5 $\int e^{ax}\,dx$

$u = ax$

$du = a\,dx$

$\frac{1}{a}du = dx$

$$\int e^{ax}\,dx = \int e^{u}\frac{1}{a}\,du =$$

$$\frac{1}{a}\int e^{u}\,du = \frac{1}{a}e^{u} + C = \boxed{\frac{1}{a}e^{ax} + C}$$

☆ Using those results:

$$\int e^{-3x}\,dx = -\frac{1}{3}e^{-3x} + C$$

$$\int \sin(\pi x)\,dx = -\frac{1}{\pi}\cos(\pi x) + C$$

$$\int \cos(97x)\,dx = \frac{1}{97}\sin(97x) + C$$

$\boxed{\text{Ex 6}}$ $\int \frac{x^2}{x^3+1}\,dx$

☆ Finding the "inside function" here is a bit harder since there isn't an as obvious outside function. Basically what we need to do is to pick u so that du appears in the integral

$$\int \frac{x^2}{x^3+1}\,dx$$

★ I'm going to intentionally pick the incorrect u so that you can see what happens.

$u = x^2$

$du = 2x\,dx$ the issue with this choice is that this doesn't help me with the x^3+1. I also don't have an extra x in the numerator to form my du.

$$\int \frac{x^2}{x^3+1}\,dx$$

☆ I'm going to do a better choice for u, but not the best choice.

$u = x^3$

$du = 3x^2\,dx$ this is better we have an x^2 in the numerator

$\frac{1}{3}du = x^2\,dx$

$$\int \frac{\frac{1}{3}du}{u+1} = \frac{1}{3}\int \frac{du}{u+1}$$ we can do another substitution here with $w = u+1$

$dw = du$

$$= \frac{1}{3}\int \frac{dw}{w} = \frac{1}{3}\ln|w| + C = \frac{1}{3}\ln|u+1| + C$$

$$= \frac{1}{3}\ln|x^3+1| + C$$

☆ The way we just did was OK but not efficient

$$\int \frac{x^2}{x^3+1}\,dx$$

Instead, $u = x^3+1$

I know the 1 will go away when I differentiate

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$$\int \frac{1}{3}\frac{du}{u} = \frac{1}{3}\ln|u| + C$$

$$= \boxed{\frac{1}{3}\ln|x^3+1| + C}$$

☆ The point of this is that sometimes you might try something that doesn't work & then you try again.

Ex 7 $\int (1+2x)\sqrt{x^2+x}\,dx$

* outside function $= \sqrt{}$
  inside function $= x^2+x$

$u = x^2+x$

$du = (2x+1)\,dx$

$\quad = (1+2x)\,dx$

$\int (1+2x)\sqrt{x^2+x}\,dx = \int \sqrt{u}\,du$

$= \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C$

$= \boxed{\frac{2}{3}(x^2+x)^{3/2} + C}$

$\boxed{\text{Ex 8}}$ $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

outside function $= e^{\square}$
inside function $= \sqrt{x}$

$u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}\,dx$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$\int e^{\sqrt{x}} \underbrace{\frac{1}{\sqrt{x}}\,dx}_{=2du} = \int 2e^{u}\,du$

$= 2e^{u} + C = \boxed{2e^{\sqrt{x}} + C}$

☆ Depending on what u is du might be part of the denominator

u-substitution with definite Integrals

$$\int_a^b f'(g(x))\,g'(x)\,dx$$

☆ Note: $a \le x \le b$

$u = g(x)$

$du = g'(x)\,dx$

$$\int f'(u)\,du$$

☆ we know $a \le x \le b$, but we don't know what u is between, but since

$u = g(x)$

$u(a) = g(a)$ and $u(b) = g(b)$

$$\int_a^b f'(g(x))\,g'(x)\,dx$$

$$= \int_{g(a)}^{g(b)} f'(u)\,du$$

$$= f(u)\Big|_{g(a)}^{g(b)}$$

$$= f(g(b)) - f(g(a))$$

☆ Not changing your endpoints can result in more work especially in calculus 3 when you have double & triple integrals.

That said, $\int_a^b f'(g(x))\,g'(x)\,dx$

$$= \int_?^? f'(u)\,du = f(u)\Big|_?^? = f(g(x))\Big|_a^b$$

$= f(g(b)) - f(g(a))$ is OK.

But, $\int_a^b f'(g(x))\, g'(x)\, dx$

$= \int_a^b f'(u)\, du$ is <u>not</u> ok.

$\boxed{\text{Ex 9}}$ $\displaystyle\int_0^1 \frac{(\tan^{-1}x)^2}{x^2+1}\, dx$

$u = x^2+1$ will result in $du = 2x\, dx$
which we don't have
instead we'll try $u = \tan^{-1}x$

$$du = \frac{1}{1+x^2}\, dx$$

$$= \frac{1}{x^2+1}\, dx$$

☆ we have a definite integral so let's change our endpoints

$$\int_0^1 \frac{(\tan^{-1}x)^2}{x^2+1}\,dx$$

$u = \tan^{-1}x$

$du = \frac{1}{1+x^2}\,dx$

$u(0) = \tan^{-1}0 = 0$

$u(1) = \tan^{-1}1 = \frac{\pi}{4}$

see the trig basics worksheet if this confuses you.

$$\int_0^1 \frac{(\tan^{-1}x)^2}{x^2+1}\,dx \qquad \frac{dx}{x^2+1} = du$$

$$= \int_0^{\pi/4} u^2\,du = \frac{1}{3}u^3 \Big|_0^{\pi/4}$$

$$= \frac{1}{3}\left(\frac{\pi}{4}\right)^3 - \frac{1}{3}0^3 = \boxed{\frac{1}{3}\left(\pi/4\right)^3}$$

☆ After we change our endpoints we don't need to worry about what u was equal to.

$\boxed{\text{Ex 10}}$ $\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin^2 x}\,dx$

☆ As per usual, we are looking for du to be in our problem. So $u = \sin^2 x$ would not be a good choice since then $du = 2\sin x \cos x\,dx$ which we don't have.

Likewise $u = \cos x$, $du = -\sin x\,dx$ is not a good choice.

Instead, $u = \sin x$
$du = \cos x\,dx$ ✓

$u(\pi/6) = \sin \pi/6 = 1/2$

$u(\pi/2) = \sin \pi/2 = 1$

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin^2 x}\,dx = \int_{1/2}^{1} \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_{1/2}^{1}$$
$$= -1 + \frac{1}{\frac{1}{2}} = -1 + 2$$
$$= \boxed{1}$$

Ex 11 $\int_1^e \frac{\sin(\ln x)}{x}\,dx$

u = inside function

$u = \ln x$

$du = \frac{1}{x}\,dx$

$u(1) = \ln 1 = 0$

$u(e) = \ln e = 1$

$$\int_1^e \sin(\ln x)\underbrace{\frac{dx}{x}}_{=du} = \int_0^1 \sin u\,du$$

$$= -\cos u\Big|_0^1 = -\cos 1 + \cos 0$$

$$= \boxed{-\cos 1 + 1}$$

cos 1 is not a value you need to know

$\boxed{\text{Ex 12}}$ $\int_0^1 x^3\sqrt{x^2+3}\,dx$

☆ This one is mildly advanced

inside function $= x^2+3$

outside function $= \sqrt{}$

$u = x^2+3$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\int_0^1 x^3\sqrt{x^2+3}\,dx = \int_0^1 x^2 x\sqrt{x^2+3}\,dx$$

$$= \int_0^1 x^2\sqrt{x^2+3}\,\underbrace{x\,dx}_{\frac{1}{2}du}$$

we know $u = x^2+3$ so $u-3 = x^2$

$$= \int_3^4 (u-3)\sqrt{u}\,\tfrac{1}{2}du = \int_3^4 \tfrac{1}{2}(u-3)u^{1/2}\,du$$

$$= \int_3^4 \tfrac{1}{2}\left[u^{3/2} - 3u^{1/2}\right]du$$

$u(0) = 0^2+3$

$u(1) = 1^2+3 = 4$

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$$\int_3^4 \frac{1}{2}\left[u^{3/2} - 3u^{1/2}\right] du$$

$$= \frac{1}{2}\left[\frac{2}{5}u^{5/2} - \cancel{3}\cdot\frac{2}{\cancel{3}}u^{3/2}\right]_3^4$$

$$= \boxed{\frac{1}{2}\left[\frac{2}{5}\cdot 4^{5/2} - 2\cdot 4^{3/2} - \left(\frac{2}{5}\cdot 3^{5/2} - 2\cdot 3^{3/2}\right)\right]}$$