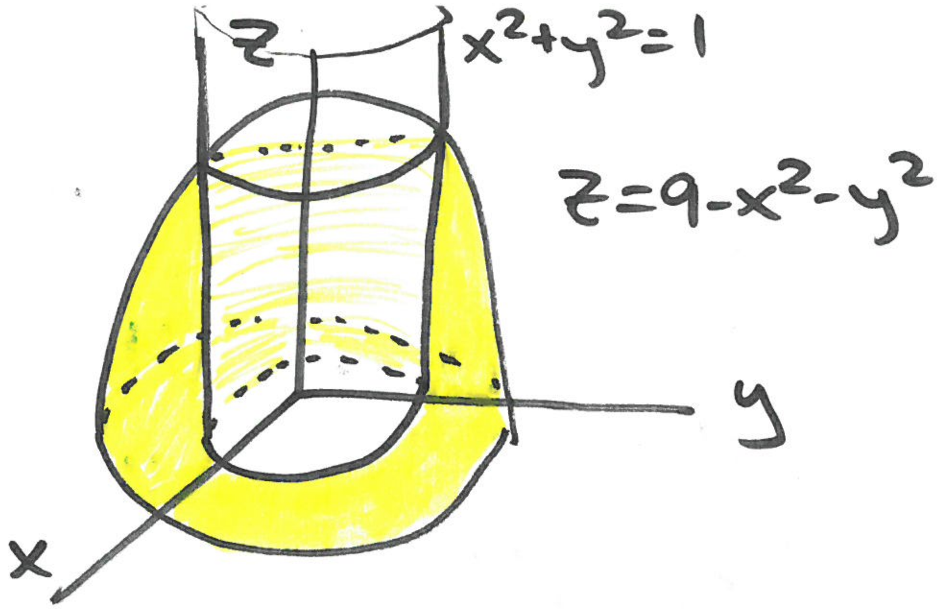
Cylindrical Coordinates

☆ Good for triple integrals that involve symmetry about an axis (ex cylinders, cones, spheres, ellipsoids, paraboloids, etc)

Examples from or inspired by *Thomas' Calculus* by Finney, Weir, & Giordano

$\boxed{\text{Ex 1}}$ Find the volume of the region bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy-plane, & lying outside the cylinder $x^2 + y^2 = 1$



☆ We could do this as a double integral with polar coordinates

$$\boxed{V=\iint_D \text{top}-\text{bottom}\; dA}$$

$$=\iint_D 9-x^2-y^2-0\; dA$$

$$=\iint 9-x^2-y^2\; dA=\int_0^{2\pi}\int_1^3 (9-r^2)\,r\,dr\,d\theta$$

$$=\ldots$$

With cylindrical coordinates:

$$\boxed{V = \iiint_F 1\,dV}$$ top surface

$$= \iint \int_0^{9-x^2-y^2} 1\,dz\,dA$$

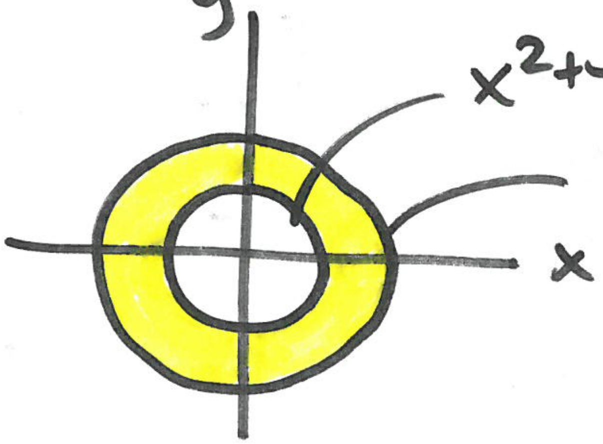
bottom surface

☆ Convert to Cylindrical Coord.

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2+y^2 &= r^2 \\ z &= z \\ dV &= dz\,r\,dr\,d\theta \end{aligned}}$$

$$V = \iint_D \int_0^{9-r^2} 1\,dz\,r\,dr\,d\theta$$

To find D project F onto the xy-plane



$x^2+y^2=1$

$x^2+y^2=9 \leftarrow$ top = bottom $9-x^2-y^2=0$

$$V=\int_0^{2\pi}\int_1^3\int_0^{9-r^2} 1\,dz\,r\,dr\,d\theta$$

whole circle centered at the origin

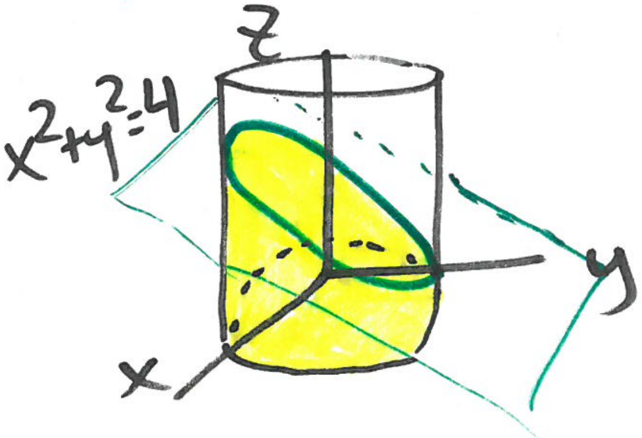
$$=\int_0^{2\pi}d\theta\int_1^3 z\Big|_0^{9-r^2} r\,dr$$

$$=2\pi\int_1^3(9-r^2)r\,dr=2\pi\int_1^3 9r-r^3\,dr$$

$$=2\pi\left(\tfrac{9}{2}r^2-\tfrac{1}{4}r^4\right)\Big|_1^3=2\pi\left(\tfrac{81}{2}-\tfrac{81}{4}-\tfrac{9}{2}+\tfrac{1}{4}\right)=32\pi$$

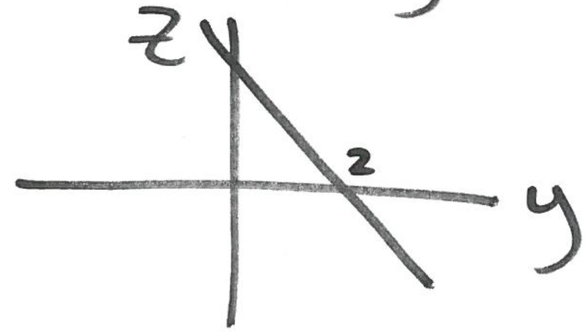
$\boxed{\text{Ex2}}$ Find the mass of the solid F bounded by the cylinder $x^2+y^2=4$ and the planes $z+y=$ and the xy-plane if $\sigma(x,y,z)=z$

$$\boxed{\text{Mass of a solid F} = \iiint_F \sigma(x,y,z)\,dV}$$



$z+y=2$

$z=2-y$



$z=2-y$

$$m=\iiint_F z\,dV$$

$$=\int_0^{2\pi}\int_0^{2}\int_0^{2-r\sin\theta} z\,dz\,r\,dr\,d\theta$$

P6

$$\int_0^{2\pi}\int_0^2 \frac{1}{2}z^2\Big|_0^{2-r\sin\theta} r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2 \frac{1}{2}(2-r\sin\theta)^2 r\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^2 (4-4r\sin\theta+r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^2 4r-4r^2\sin\theta+r^3\sin^2\theta\,dr\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi} 2r^2-\frac{4}{3}r^3\sin\theta+\frac{1}{4}r^4\sin^2\theta\Big|_0^2\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi} 8-\frac{32}{3}\sin\theta+4\sin^2\theta\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi} 8-\frac{32}{3}\sin\theta+\frac{4}{2}(1-\cos 2\theta)\,d\theta$$

$$=\frac{1}{2}\left[8\theta+\frac{32}{3}\cos\theta+2\left(\theta-\frac{1}{2}\sin 2\theta\right)\right]$$

$\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$

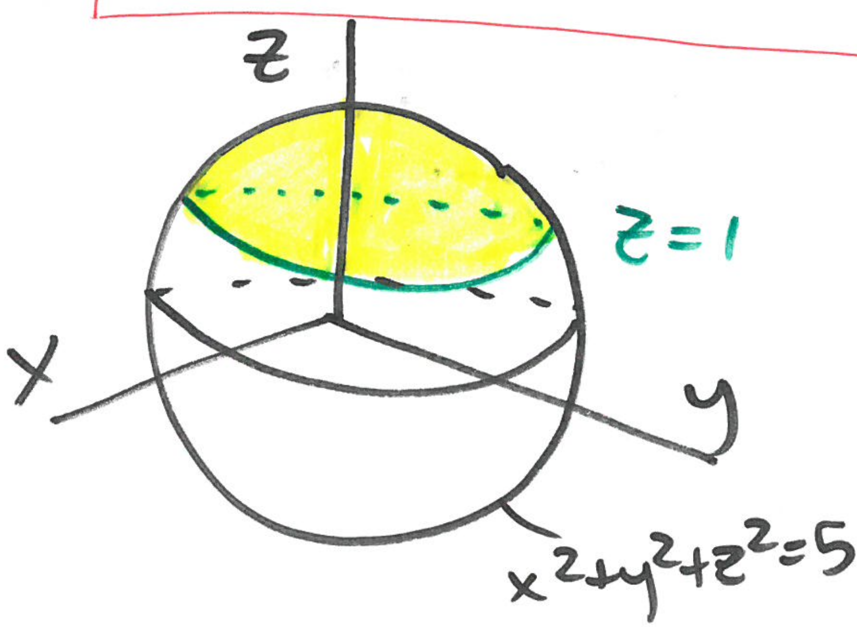
$\frac{1}{2}$ angle identity. would be given on a test

$$=\frac{1}{2}\left[16\pi+\cancel{\frac{32}{3}}+2(2\pi-0)-\cancel{\frac{32}{3}}\right]$$

$$=\boxed{10\pi}$$

Ex 3 Find the average value of $f(x,y,z)=\sqrt{9x^2+9y^2}\,z$ over the solid F bounded on top by the sphere $x^2+y^2+z^2=5$ and below by the plane $z=1$.

$$\text{Average value of } f(x,y,z) \text{ over } F = \frac{\iiint_F f(x,y,z)\,dV}{\iiint_F 1\,dV}$$



$f = \sqrt{9x^2+9y^2}\, z$ P8

$= \sqrt{9r^2}\, z = 3rz$

With cylindrical:

$$\text{Average Value} = \frac{\int_0^{2\pi}\int_0^{2}\int_1^{\sqrt{5-r^2}} (3rz)\,dz\,r\,dr\,d\theta}{\int_0^{2\pi}\int_0^{2}\int_1^{\sqrt{5-r^2}} 1\,dz\,r\,dr\,d\theta}$$

$x^2+y^2+z^2=5$

$z=\sqrt{5-x^2-y^2}$

$=\sqrt{5-r^2}$

top = bottom

$\sqrt{5-r^2} = 1$

$5-r^2 = 1^2$

$4 = r^2$

$r = 2$

If you enjoy suffering, with spherical:

$$\text{Average Value} = \frac{\displaystyle\int_0^{2\pi}\int_0^{\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)}\int_{\frac{1}{\cos\phi}}^{\sqrt{5}} \underbrace{3\rho\sin\phi}_{3r}\,\underbrace{\rho\cos\phi}_{z}\,\rho^2\sin\phi \, d\rho\, d\phi\, d\theta}{\displaystyle\int_0^{2\pi}\int_0^{\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)}\int_{\frac{1}{\cos\phi}}^{\sqrt{5}} 1\,\underbrace{\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}_{dV}}$$

$$\sqrt{5} = \frac{1}{\cos\phi}$$

$$\cos\phi = \frac{1}{\sqrt{5}}$$

$$\phi = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$z = 1$$

$$\rho\cos\phi = 1$$

$$\rho = \frac{1}{\cos\phi}$$