Lines in 3D

Recall: In class we derived two equations for a line.

vector equation: $\vec{r}(t) = \vec{r_0} + \vec{v}t$

where $\vec{r_0}$ is a point on the line written as a vector & $\vec{v}$ is the direction vector of the line (this is our slope)

Parametric equations:

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

(x_0, y_0, z_0) is a point on our line

& $\vec{v} = \langle a, b, c \rangle$ is our direction vector

Examples from *Calculus for Scientists & Engineers* by Briggs/Cochran/Gillett

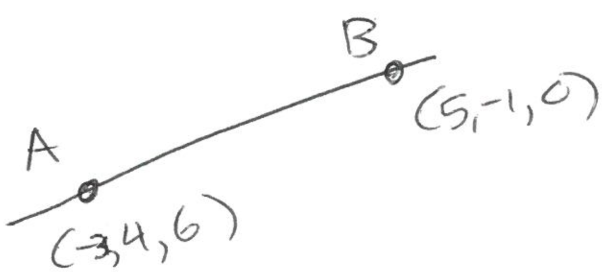
Ex1 Find a vector eqn of the line through $(-3, 2, -1)$ in the direction of vector $\vec{v} = \langle 1, -2, 0 \rangle$

☆ to find the eqn of a line we need a pt & the direction vector. they've given us both.

$$\underbrace{\vec{r}(t)}_{\text{don't forget this part; it isn't an equation unless it is equal to something}} = \underbrace{\langle -3,2,-1\rangle}_{\text{pt written as a vector}} + \underbrace{\langle 1,-2,0\rangle}_{\text{direction vector}} t$$

☆ If they wanted parametric equations: $x = -3 + 1t$
$y = 2 - 2t$
$z = -1 + 0t$

$\boxed{\text{Ex 2}}$ Find parametric equations of the line through $(-3,4,6)$ & $(5,-1,0)$



☆ We need a point & our direction vector. They gave us 2 points, either will work. To get our direction vector, we form a vector from one point to another.

$$\overrightarrow{AB} = \langle 5-(-3), -1-4, 0-6\rangle = \langle 8,-5,-6\rangle$$

This vector is on the line so it is parallel to the line, which is what we want.

$$x = x_0 + at \qquad \boxed{\begin{aligned} x &= -3+8t \\ y &= 4-5t \\ z &= 6-6t \end{aligned}}$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

☆ Note: we could have found $\overrightarrow{BA}$ and/or used point B and that would have given us an equivalent answer.

So $\begin{aligned} x &= 5+8t \\ y &= -1-4t \\ z &= 0-6t \end{aligned}$ is also good

Ex 3 This is just a variation of the previous problem. Find parametric eqns for line segment from $(-3,4,6)$ to $(5,-1,0)$

* We can use our previous answer if we specify bounds for t.

$$\boxed{\begin{aligned} x &= -3+8t \\ y &= 4-5t \qquad 0 \le t \le 1 \\ z &= 6-6t \end{aligned}}$$

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Note: $x = -3 + 8t$
$y = 4 - 5t$
$z = 6 - 6t$

We want our line segment to start at $(-3,4,6)$ so we just need to find the t value that will give us that point

$-3 = -3 + 8t \rightarrow t = 0$ (the same t value will work for all of them if you found the correct eqn.)
$4 = 4 - 5t$
$6 = 6 - 6t$

We want our line segment to finish at $(5,-1,0)$ so

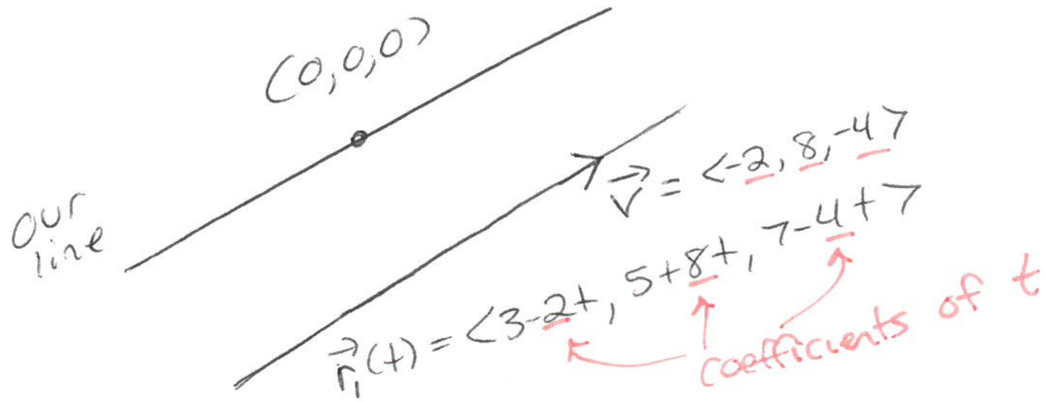
$5 = -3 + 8t \rightarrow 5 + 3 = 8t \quad 8 = 8t \rightarrow t = 1$
$-1 = 4 - 5t$
$0 = 6 - 6t$

☆ If we had used $x = 5 + 8t$
$y = -1 - 4t$
$z = 0 - 6t$) $-1 \leq t \leq 0$

$-3 = 5 + 8t$
$4 = -1 - 4t$
$6 = -6t \rightarrow t = -1$ (you don't need to show how you find the t bounds; you just need to find them)

$5 = 5 + 8t$
$-1 = -1 - 4t$
$0 = -6t \rightarrow t = 0$

Ex 4 Find a vector equation of the line through $(0,0,0)$ that is parallel to the line $\vec{r}_1(t) = \langle 3-2t, 5+8t, 7-4t \rangle$

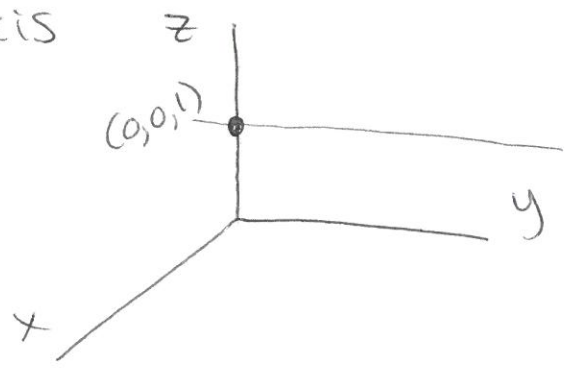


☆ We need a point on our line & a vector in the direction of our line (or parallel to our line). Since $\vec{r}(t)$ is parallel to our line, its direction vector is parallel to ours.

$$\vec{r}(t) = \langle 0,0,0 \rangle + \langle -2, 8, -4 \rangle t$$

$$\boxed{\vec{r}(t) = \langle -2t, 8t, -4t \rangle}$$

Ex 5 Find a vector equation of the line through $(0,0,1)$ that is parallel to the y-axis



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A point on the y-axis has coordinates $(0,y,0)$. We can represent that as the line with parametric equations $x=0, z=0, y=t$ ← y can be anything or as a vector equation: $\vec{r}(t) = \langle 0,1,0 \rangle t$

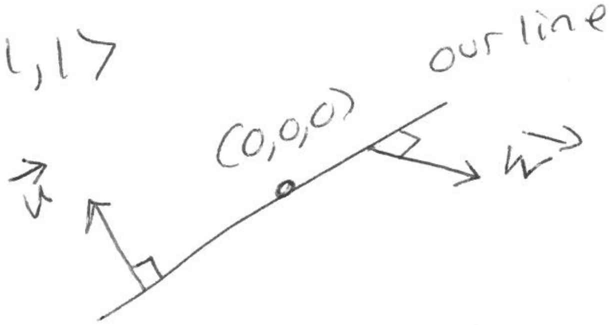
vector equation for y-axis

I used the point $(0,0,0)$, but any point with x & z equal to zero would have worked

$$\underbrace{\vec{r}(t)}_{\text{our line}} = \underbrace{\langle 0,0,1 \rangle}_{\text{pt on our line}} + \langle 0,1,0 \rangle t$$

note: $\vec{v}$ is equivalent to $\hat{j}$

Ex 6 Find parametric equations of the line through $(0,0,0)$ that is perpendicular to both $\vec{u} = \langle 1,0,2 \rangle$ & $\vec{w} = \langle 0,1,1 \rangle$



☆ Again we have a point, but still need our direction vector $\vec{v}$.

$\vec{u}$ is $\perp$ to our line so it is $\perp$ to $\vec{v}$

$\vec{w}$ is $\perp$ to our line so $\vec{w}$ is $\perp$ to $\vec{v}$

$\vec{v}$ is a vector $\perp$ to both $\vec{u}$ & $\vec{w}$

so $\vec{v} = \vec{u} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix}$

$= \langle 0-2, -(1-0), 1-0 \rangle = \langle -2, -1, 1 \rangle$

$$\boxed{\begin{matrix} x = 0 - 2t \\ y = 0 - 1t \\ z = 0 + 1t \end{matrix} \quad \text{or} \quad \begin{matrix} x = -2t \\ y = -t \\ z = t \end{matrix}}$$

Ex 7 Find a vector equation of the line through $(-2,5,3)$ that is perpendicular to both $\vec{u} = \langle 1,1,2 \rangle$ and the x-axis

This is similar to the last problem, but this time

$\vec{v} = \vec{u} \times \langle 1,0,0 \rangle$

direction vector of the x-axis, $\hat{i}$

$$\vec{v} = \vec{u} \times \langle 1,0,0 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \langle 0-0, -(0-2), 0-1 \rangle$$

$$= \langle 0, 2, -1 \rangle$$

$$\boxed{\vec{r}(t) = \langle -2,5,3 \rangle + \langle 0,2,-1 \rangle t}$$

Ex 8 Determine parametric equations for the line that is perpendicular to lines L_1: $x=4t$, $y=1+2t$, $z=3t$ and L_2: $x=-1+s$, $y=-7+2s$, $-12+3s$ and passes through their point of intersection.

☆ If they intersect, then they have a point in common. I'm going to set the x, y, & z components equal and solve for t & s.

$$\begin{array}{ll} L_1 & L_2 \\ 4t & = -1+s \\ 1+2t & = -7+2s \\ 3t & = -12+3s \end{array}$$

$4t = -1 + s$

$\xrightarrow{\text{mult by } -2}$ $-2-4t = 14-4s$

$-2 = 13 - 3s$

$-15 = -3s$

$s = 5$

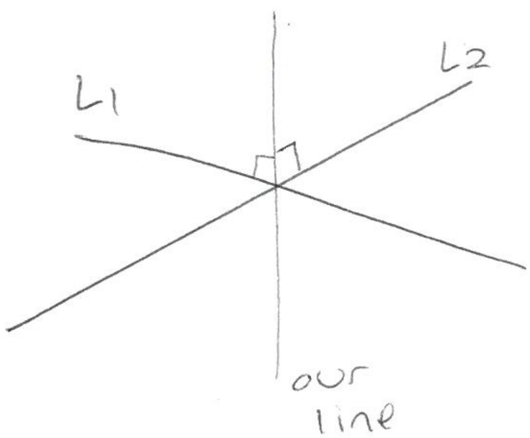
☆ Solving for t or s & plugging in works too

Since $4t = -1 + s$ & $s = 5$

$$4t = -1 + 5$$
$$4t = 4 \rightarrow t = 1$$

Check the 3rd equation to double check your values.

$$3t = -12 + 3s$$
$$3 \cdot 1 \stackrel{?}{=} -12 + 3 \cdot 5 = -12 + 15 = 3 \checkmark$$



L_1 when $t = 1$:

$x = 4 \cdot 1 = 4$
$y = 1 + 2 \cdot 1 = 3$
$z = 3 \cdot 1 = 3$
point $= (4, 3, 3)$

☆ At this point, the problem becomes like $\boxed{\text{Ex 6}}$, $\vec{V} = \vec{V}_{L_1} \times \vec{V}_{L_2}$ since $\vec{V}_{L_1}$ is $\perp$ to $\vec{V}$ & $\vec{V}_{L_2}$ is $\perp$ to $\vec{V}$

$$\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

coefficients of t in L_1

coefficients of s in L_2

$$= \langle 6 - 6, -(12 - 3), 8 - 2 \rangle$$
$$= \langle 0, -9, 6 \rangle$$

$$\boxed{\begin{aligned} x &= 4 + \cancel{0}t \\ y &= 3 - 9t \\ z &= 3 + 6t \end{aligned}}$$

$\boxed{\text{Ex 9}}$ Determine if the lines are parallel, intersecting, or skew. ← neither parallel nor intersecting
If they intersect find their point of intersection.

$$\vec{r}_1(t) = \langle 1+6t, 3-7t, 2+t \rangle$$

$$\vec{r}_2(s) = \langle 10+3s, 6+s, 14+4s \rangle$$

☆ First find their direction vectors

$\vec{v}_1 = \langle 6, -7, 1 \rangle$ $\quad \vec{v}_2 = \langle 3, 1, 4 \rangle$

we can tell they aren't parallel since $\vec{v}_1 \neq c\vec{v}_2$ for any constant c.
Alternatively, you could show $\vec{v}_1 \times \vec{v}_2 \neq \vec{0}$

☆ Next set them equal, if they intersect they will have a point in common. If they don't we'll get a contradiction

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$\vec{r}_1 \quad \vec{r}_2$

$1+6t = 10+3s \longrightarrow 1+6t=10+3s$

$3-7t = 6+s$

$2+t = 14+4s \xrightarrow{\text{mult by} -6} -12-6t=-84-24s$

$-11=-74-21s$

$63=-21s$

$s=-3$

☆ You could actually add all the eqns & solve for s faster

If $s=-3$, $1+6t=10+3s \rightarrow$

$1+6t=10+3(-3)=1$

$6t=0$

$t=0$

Check with the 3rd equation to see if it works:

$3-7t=6+s$

$3-7\cdot 0 \stackrel{?}{=} 6-3$

$3=3$ ✓

They do intersect. To find the point of intersection find either $\vec{r}_1(0)$ or $\vec{r}_2(-3)$

$\vec{r}_1(0) = \langle 1,3,2 \rangle$ as a point, $\boxed{(1,3,2)}$

Ex 10 Same directions as Ex 9

$\vec{r_1}(t) = \langle 4+5t, 2t, 1+3t \rangle$

$\vec{r_2}(s) = \langle 10s, 6+4s, 4+6s \rangle$

☆ First check for parallel

$\vec{v_1} = \langle 5, 2, 3 \rangle$ $\vec{v_2} = \langle 10, 4, 6 \rangle$

$2\vec{v_1} = \vec{v_2}$ so $\vec{r_1}$ & $\vec{r_2}$ are parallel.

Ex 11 Same directions as Ex 9

$\vec{r_1}(t) = \langle 4, 6-t, 1+t \rangle$

$\vec{r_2}(s) = \langle -3-7s, 1+4s, 4-s \rangle$

☆ Check for parallel

$\vec{v_1} = \langle 0, -1, 1 \rangle$ $\vec{v_2} = \langle -7, 4, -1 \rangle$

$\vec{v_1} \neq c\vec{v_2}$ for any c

$\longrightarrow$ not parallel

☆ Check for intersecting

$\vec{r}_1 \quad \vec{r}_2$

$4 = -3 - 7s \rightarrow 7 = -7s \rightarrow s = -1$

$6 - t = 1 + 4s$ if $s = -1$: $6 - t = 1 - 4 \rightarrow t = 9$

$1 + t = 4 - s$

☆ Check if it works: $1 + t = 4 - s$

$1 + 9 = 4 - (-1)$

$10 = 5$

$\#!$ contradiction

☆ Since they aren't parallel or intersecting, they are $\boxed{\text{skew}}$

$\boxed{\text{Ex 12}}$ Find the point (if it exists) at which the following plane & line intersects

$$z = -8, \quad \vec{r} = \langle 3t - 2, t - 6, -2t + 4 \rangle$$

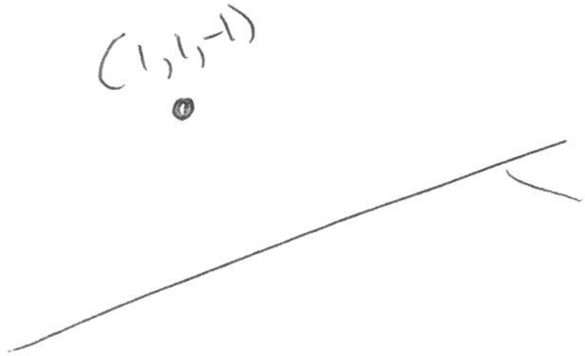
☆ If they intersect then the z value of the line must be −8 for some t.

$-8 = -2t + 4$

$-12 = -2t \rightarrow t = 6 \qquad \vec{r}(6) = \langle 18 - 2, 0, -8 \rangle$

$\boxed{(16, 0, -8)}$

$\boxed{\text{Ex 13}}$ Find the distance from a point $(1,1,-1)$ and the line with direction vector $\langle -6,8,3\rangle$ through the origin.



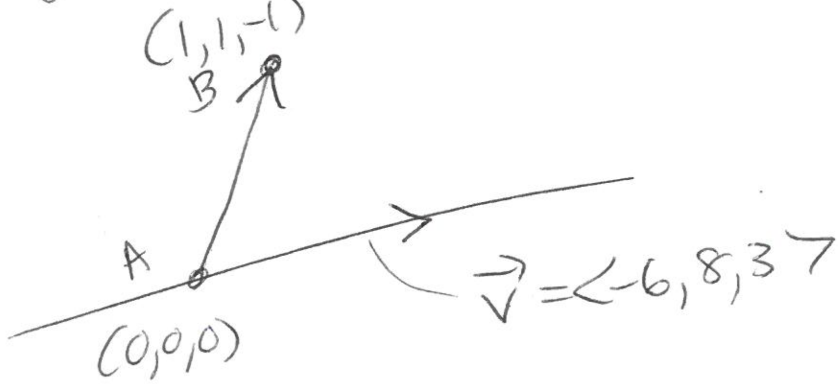
$$\vec{r}(t) = \langle 0,0,0\rangle + \langle -6,8,3\rangle t$$
$$= \langle -6,8,3\rangle t$$

similar to $\boxed{\text{Ex1}}$

☆ There is a formula for this in your book that you <u>don't</u> need to memorize. It is more satisfying to do this with vector & scalar projections

☆ Find a point on the line. In this case $(0,0,0)$

☆ Form a vector from the line to the given point.



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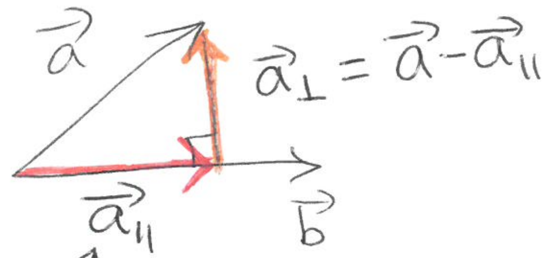
$\overrightarrow{AB} = \langle 1-0, 1-0, -1-0 \rangle = \langle 1, 1, -1 \rangle$

(that should make sense since for this problem $\overrightarrow{AB}$ goes from the origin to point $(1,1,-1)$)

From 1.3:



$\text{proj}_{\vec{b}}\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|} \dfrac{\vec{b}}{\|\vec{b}\|}$

☆ We want $\|\vec{a}_\perp\|$

For this problem $\vec{a} = \overrightarrow{AB}$ $\quad \vec{b} = \vec{v}$

$$\vec{a}_\parallel = \frac{\langle 1,1,-1 \rangle \cdot \langle -6,8,3 \rangle}{\sqrt{6^2+8^2+3^2}} \frac{\langle -6,8,3 \rangle}{\sqrt{6^2+8^2+3^2}}$$

$$= \frac{-6+8-3}{(109)} \langle -6,8,3 \rangle$$

$$= \frac{\langle 6,-8,-3 \rangle}{109}$$

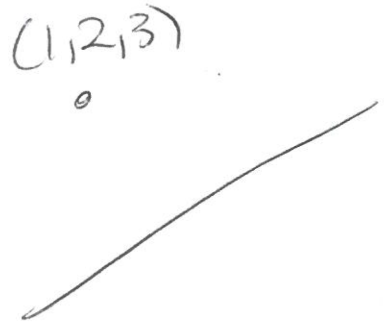
$$\vec{a}_\perp = \vec{a} - \vec{a}_\parallel = \langle 1,1,-1 \rangle - \left\langle \frac{6}{109}, \frac{-8}{109}, \frac{-3}{109} \right\rangle$$

$$= \left\langle \frac{103}{109}, \frac{117}{109}, \frac{-106}{109} \right\rangle$$

$$d = \|\vec{a}_{\perp}\| = \frac{\sqrt{103^2 + 117^2 + 106^2}}{109}$$

$\boxed{\text{Ex 14}}$ Find the distance from the point $(1,2,3)$ to the line

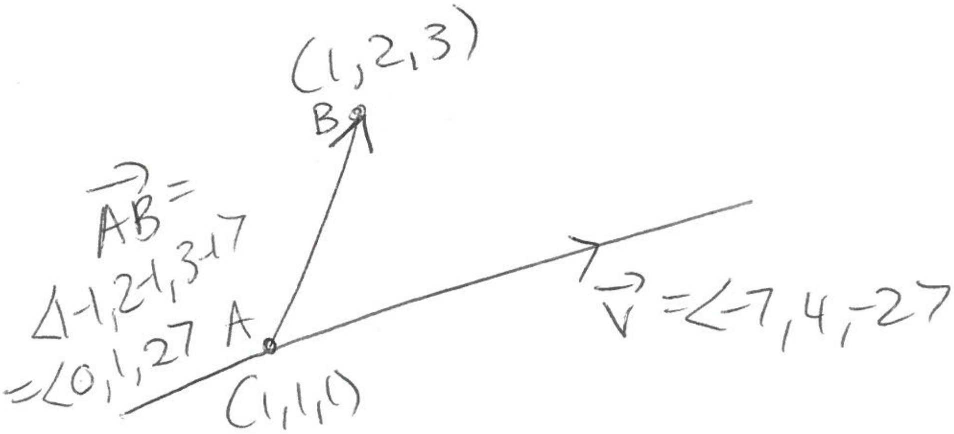
$\vec{r}(t) = \langle 1,1,1 \rangle + \langle -7,4,-2 \rangle t$



☆ I just made this one up. It is similar to your silly webassign problem

☆ Find a point on the line. Any point will do, but I'll choose $t=0$,

$\vec{r}(0) = \langle 1,1,1 \rangle$ so $(1,1,1)$



$$\vec{a}_{\parallel} = \frac{\vec{AB}\cdot\vec{v}}{\|\vec{v}\|}\,\frac{\vec{v}}{\|\vec{v}\|}$$

$$= \frac{\langle 0,1,2\rangle \cdot \langle -7,4,-2\rangle}{\sqrt{7^2+4^2+2^2}}\,\frac{\langle -7,4,-2\rangle}{\sqrt{7^2+4^2+2^2}}$$

$$= \frac{0+4-4\,\langle -7,4,-2\rangle}{7^2+4^2+2^2}$$

$$= \vec{0}$$

☆ Since $\vec{a}_{\parallel}$, the portion of $\vec{a}$ parallel to $\vec{v}$, is $\vec{0}$ that implies $\vec{AB}$ is $\perp$ to $\vec{v}$ or $\vec{a}_{\perp} = \vec{AB}$

$$d = \|\vec{AB}\| = \sqrt{0^2+1^2+2^2} = \boxed{\sqrt{5}}$$

☆ This is silly since our point on the line just happens to be in the perfect spot.

our picture now that we know $\vec{AB} \perp$ to $\vec{v}$