Planes

In class we derived $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ where (x_0, y_0, z_0) is a point on the plane and $\vec{n}=\langle a,b,c\rangle$ is a vector perpendicular to the plane called the normal vector.

Sometimes the plane equation is simplified to $ax+by+cz=d$. We can see the coefficients of x, y, & z will give us our normal vector.

Ex 1 Find an equation of the plane that passes through the point $(1,0,-3)$ with normal vector $\vec{n}=\langle 1,-1,2\rangle$.

☆ They've given us both of the things we need.

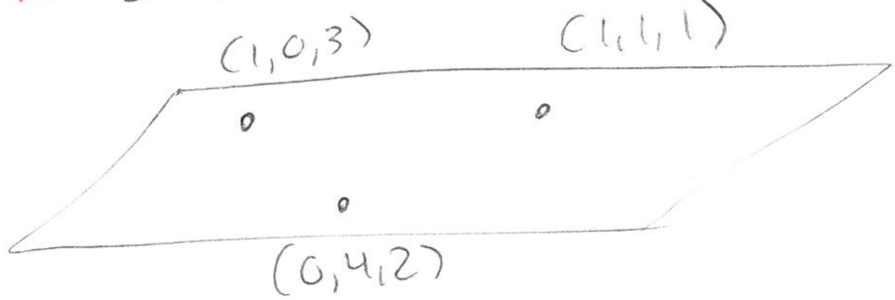
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$\boxed{1(x-1)-1(y-0)+2(z+3)=0}$$

Don't forget to write this!

Ex 2 Find an equation of the plane passing through the points $(1,0,3)$, $(0,4,2)$, and $(1,1,1)$

☆ Draw a picture, but don't plot the points.



☆ We know the points aren't all on the same line because if they were this question wouldn't have a distinct answer.

☆ We need 2 pieces of informatio to find an equation of a plane: a point which we have and a vector $\vec{n}$ perpendicular to the plane.

☆ A general strategy for planes is to find 2 vectors contained in the plane or parallel to the plane & then take their cross product

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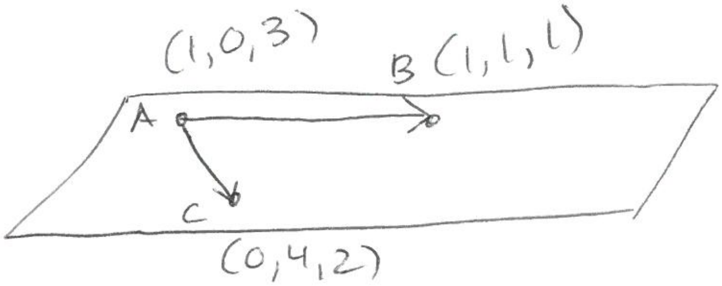


$$\overrightarrow{AB} = \langle 1-1, 1-0, 1-3 \rangle = \langle 0, 1, -2 \rangle$$

$$\overrightarrow{AC} = \langle 0-1, 4-0, 2-3 \rangle = \langle -1, 4, -1 \rangle$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -2 \\ -1 & 4 & -1 \end{vmatrix}$$

$$= \langle -1+8, -(0-2), 0+1 \rangle$$

$$= \langle 7, 2, 1 \rangle$$

☆ We could have done $\overrightarrow{CB} \times \overrightarrow{AC}$ or any other combination of 2 vectors in our plane. Basically our normal vector could be any nonzero constant multiple of $\langle 7, 2, 1 \rangle$

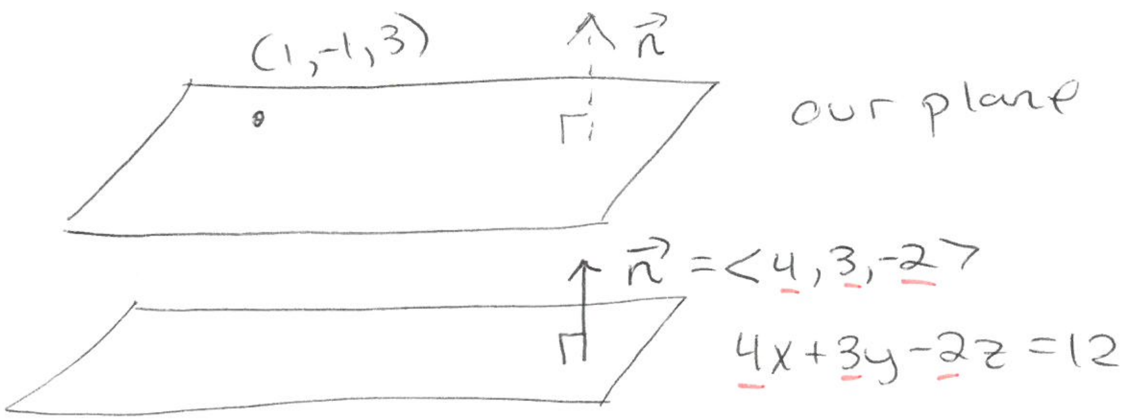
$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\boxed{7(x-1) + 2(y-0) + 1(z-3) = 0}$$

☆ I chose the point A for my (x_0, y_0, z_0), but any of the 3 points would also work. The equations of the plane simplified would give us the same answer.

Ex3 Find an equation of the plane parallel to the plane $4x+3y-2z=12$ and containing the point $(1,-1,3)$

☆ Draw a picture



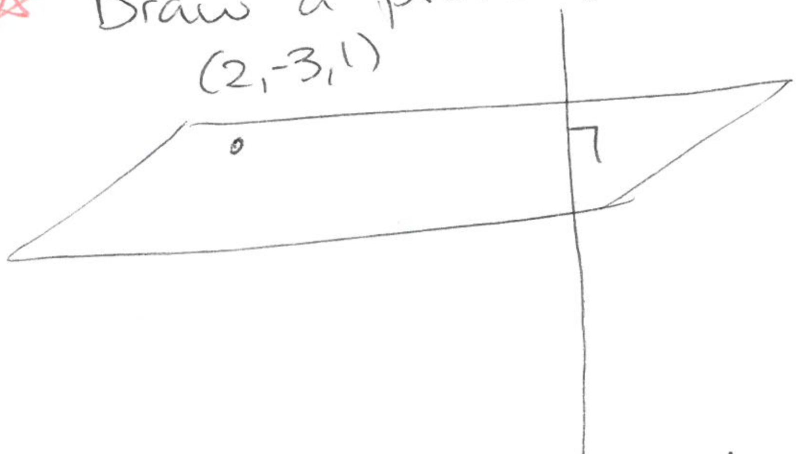
☆ We need a point on our plane & a vector perpendicular to our plane. Notice that They gave us the point. Since our plane is parallel to another plane that plane's normal vector is also $\perp$ to our plane

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$\boxed{4(x-1)+3(y+1)-2(z-3)=0}$$

Ex 4 Find an equation of the plane passing through $(2,-3,1)$ normal to the line $\vec{r}(t) = \langle 2+t,\ 3t,\ 2-3t \rangle$

☆ Draw a picture



☆ Since our line is $\perp$ to our plane we can use the line's direction vector as the plane's normal vector

$\vec{v} = \langle 1, 3, -3 \rangle$ (Recall: these are the coefficients of t)

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\boxed{1(x-2) + 3(y+3) - 3(z-1) = 0}$$

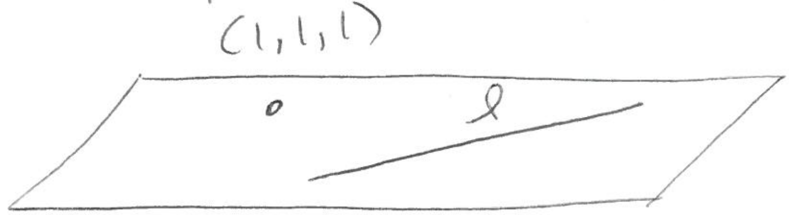
Ex 5 Find a vector equation of the line through $(2,-3,1)$ and normal to the plane we found in Ex 4

$$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + \underbrace{\langle a, b, c \rangle}\, t$$

Be careful; for lines $\langle a,b,c \rangle$ is the direction vector

$$\boxed{\vec{r}(t) = \langle 2,-3,1 \rangle + \langle 1,3,-3 \rangle t}$$

$\boxed{\text{Ex 6}}$ Find an equation of the plane through $(1,1,1)$ that intersects the xy-plane in the same line as does the plane $3x+2y-z=6$

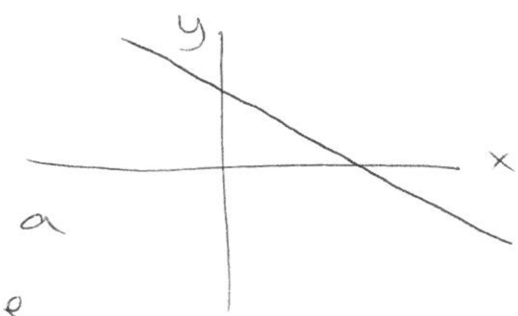


☆ Need to find the line where $3x+2y-z=6$ intersects the xy-plane.

xy-plane $\rightarrow z=0$ so $3x+2y-0=6$

$$y = \frac{6-3x}{2} = 3-\frac{3}{2}x$$

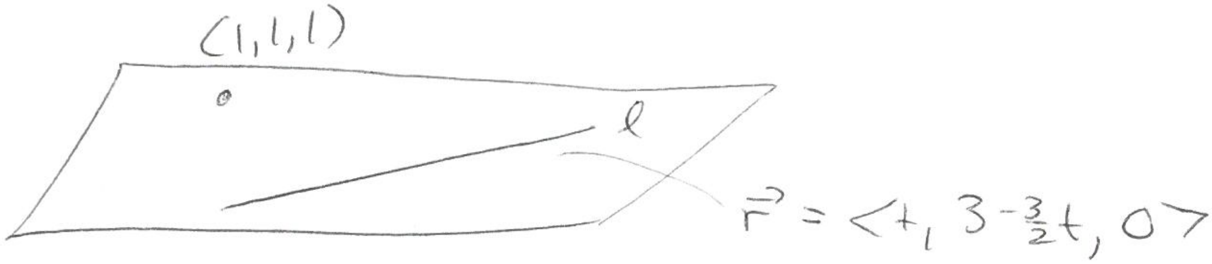
2D equation of a line in xy-plane



As a 3D line: $\vec{r}(t) = \langle t,\ 3-\frac{3}{2}t,\ 0 \rangle$

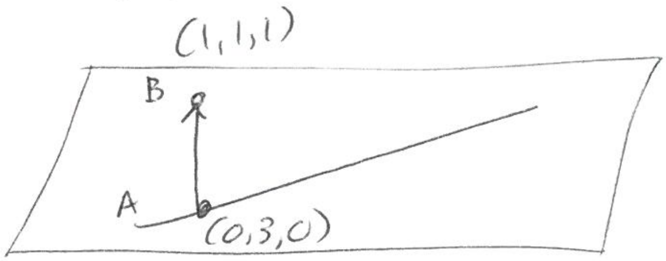
☆ Note: $x=t$ gives $y=$ xy-plane

☆ Now this becomes the relatively simple problem of finding the eqn of a plane containing a point & a line



☆ We want to find 2 vectors in our plane. If we cross them, we will get a vector $\perp$ to our plano.

☆ First find a point on the line by setting t equal to any number, I'm going to do $t=0$. $\vec{r}(0) = \langle 0, 3, 0 \rangle$ so $(0,3,0)$ is a pt on our line



☆ Find $\overrightarrow{AB}$ to get a vector in our plane

$$\overrightarrow{AB} = \langle 1-0, 1-3, 1-0 \rangle = \langle 1, -2, 1 \rangle$$

☆ To get a 2nd vector in our plane either find another point on our line & form a vector from A to that point or use $\vec{v} = \langle 1, -\frac{3}{2}, 0 \rangle$ the direction vector for our line. I'm going to use $2\vec{v} = \langle 2, -3, 0 \rangle$ which is also parallel to our plane, but easier to work with.

$\vec{n} = \overrightarrow{AB} \times 2\vec{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 2 & -3 & 0 \end{vmatrix} = \langle 0+3, -(0-2), -3+4 \rangle$$

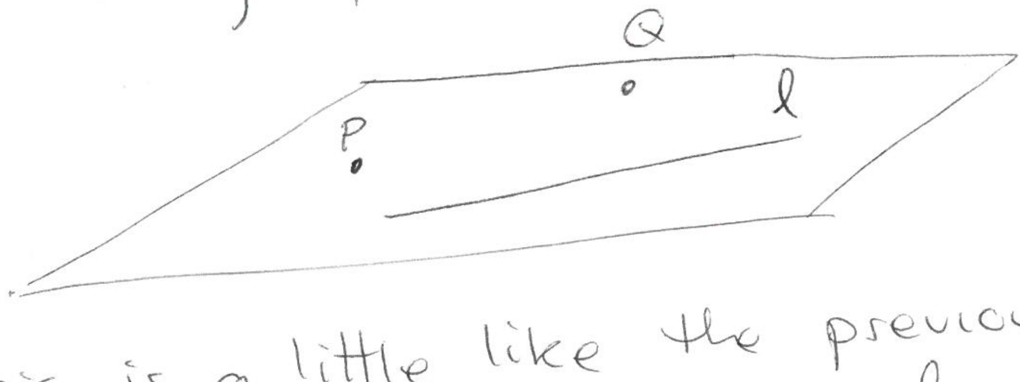
$$= \langle 3, 2, 1 \rangle$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

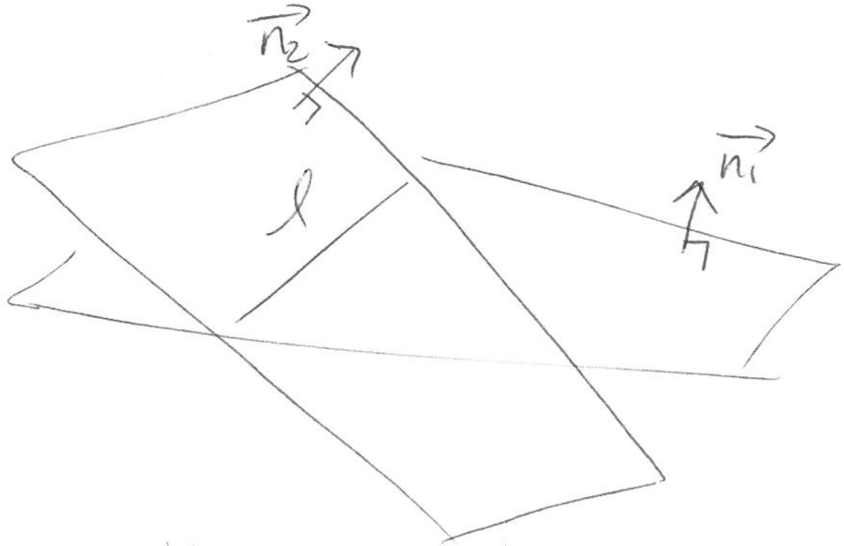
$$\boxed{3(x-1) + 2(y-1) + 1(z-1) = 0}$$

☆ We could have used point (0,3,0) instead. When we simplify it would simplify to the same thing as the plane we found.

Ex 7 Find an equation of the plane that passes through points $P(1,0,-1)$ & $Q(2,1,0)$ & is parallel to the line of intersection of the planes $x+y+z=5$ and $3x-y=4$



☆ This is a little like the previous problem. Let's first focus on l.



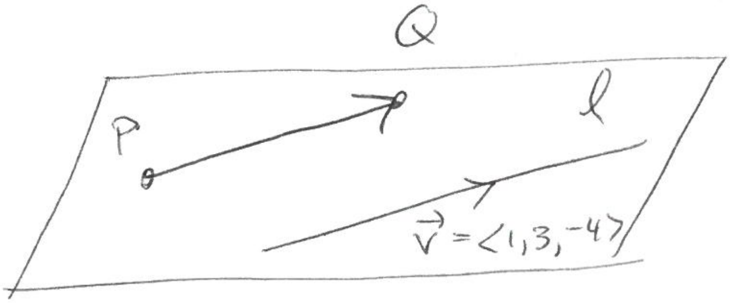
$\vec{n_1}$ is $\perp$ to everything in its plane so $\vec{n_1}$ is $\perp$ to ℓ & therefore $\perp$ to $\vec{v}$ our line's direction vector.

Likewise, $\vec{n_2}$ is $\perp$ to $\vec{v}$.

Since $\vec{v}$ is $\perp$ to both $\vec{n_1}$ & $\vec{n_2}$,

$$\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{array}{c l} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix} & \begin{array}{l} \\ \leftarrow x+y+z=5 \\ \leftarrow 3x-y=4 \end{array} \end{array}$$

$$= \langle 0+1, -(0-3), -1-3 \rangle = \langle 1, 3, -4 \rangle$$



($\vec{PQ}$ & $\vec{v}$ look vaguely parallel here -- they aren't actually or $\vec{PQ} \times \vec{v} = \vec{0}$)

☆ Once we find $\vec{PQ}$ we'll have 2 vectors contained in our plane. To find $\vec{n}$ we will cross them

$\overrightarrow{PQ} = \langle 2-1, 1-0, 0-(-1) \rangle = \langle 1,1,1 \rangle$

$$\vec{n} = \overrightarrow{PQ} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 3 & -4 \end{vmatrix}$$

$$= \langle -4-3, -(-4-1), 3-1 \rangle = \langle -7, 5, 2 \rangle$$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$$\boxed{-7(x-2) + 5(y-1) + 2(z-0) = 0}$$

☆ I used point $Q(2,1,0)$. P would also work

Ex 8 Find parametric equations of the line of intersection from Ex 7

☆ We already found $\vec{v} = \langle 1,3,-4 \rangle$. All we need is a point on the line.

$x+y+z=5 \qquad 3x-y=4$

☆ To find a point on the line of intersection of 2 planes, fix one variable as 0 & solve for the other 2. If that variable is never zero, you will get a contradiction. Just fix a different variable to 0.

11 $x+y+z=5 \qquad 3x-y=4$

☆ I'm going to set $x=0$, since $3x$ might make things harder.

$0+y+z=5 \qquad 3\cdot 0-y=4$

$-y=4$

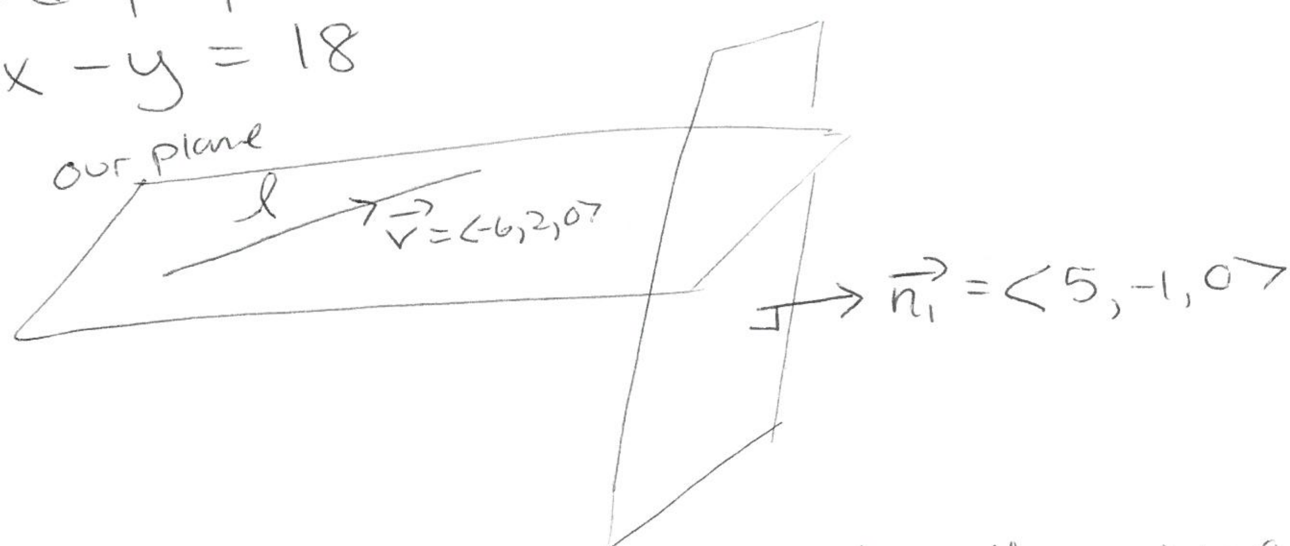
$y=-4 \rightarrow -4+z=5 \qquad (y+z=5)$

$z=9$

point: $(0,-4,9)$

$$\begin{aligned} x&=x_0+at \\ y&=y_0+bt \\ z&=z_0+ct \end{aligned} \longrightarrow \boxed{\begin{aligned} x&=0+1t \\ y&=-4+3t \\ z&=9-4t \end{aligned}}$$

Ex9 Find an equation of the plane containing the line $x=4-6t$, $y=2t$, $z=3$ and perpendicular to the plane $5x-y=18$



$\vec{v}$ & $\vec{n_1}$ are both parallel to the plane we want so $\vec{n}=\vec{v}\times\vec{n_1}$

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$$\vec{n} = \vec{v} \times \vec{n}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & 2 & 0 \\ 5 & -1 & 0 \end{vmatrix}$$

$$= \langle 0-0, -(0-0), 6-10 \rangle$$

$$= \langle 0, 0, -4 \rangle$$

☆ To find a point on our plane, we just need a point on our line. Set $t=0$ to get $x = 4-6\cdot 0 = 4$, $y = 2\cdot 0 = 0$, $z=3$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$0(x-4) + 0(y-0) - 4(z-3) = 0$$

$$\boxed{-4(z-3) = 0} \quad \text{or } z=3$$

Ex 10 The angle between 2 planes is the angle θ between the normal vectors of the planes where the directions of the normal vectors are chosen so that $0 \le \theta < \pi$. Find the angle between $5x + 2y - z = 0$ & $-3x + y + 2z = 0$

☆ Recall from 1.3: $\vec{a} \circ \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$

$$\vec{n}_1 \circ \vec{n}_2 = \|\vec{n}_1\| \, \|\vec{n}_2\| \cos\theta$$

$$\langle 5,2,-1 \rangle \circ \langle -3,1,2 \rangle = \sqrt{5^2+2^2+1^2} \sqrt{3^2+1^2+2^2} \cos\theta$$

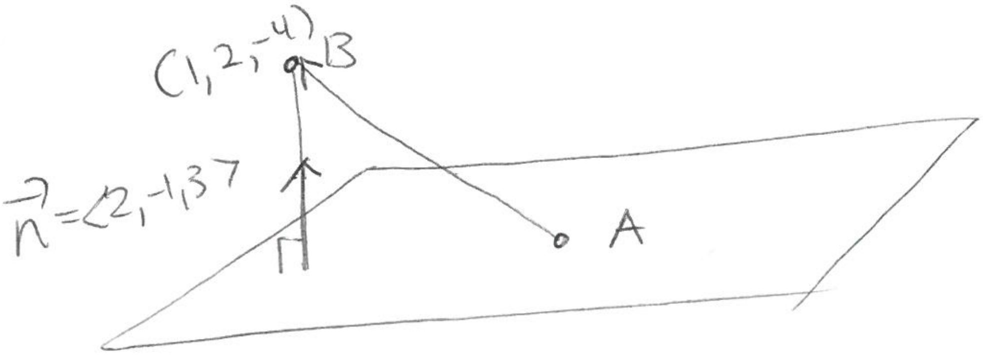
$$-15 + 2 - 2 = \sqrt{30} \sqrt{14} \cos\theta$$

$$\frac{-15}{\sqrt{30}\,\sqrt{14}} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{-15}{\sqrt{30}\,\sqrt{14}}\right) \approx 2.39 \text{ radians}$$

☆ If our angle was bigger than $\pi/2$, then we say the angle between the planes is $\pi - \theta$

Ex 11 Find the distance from the point $(1,2,-4)$ to the plane $2x-y+3z=1$



☆ Your book gives an equation for this, but we can just think about it instead. We can find a point on the plane $2x-y+3z=1$. We are looking for any point that will work so $x=0, z=0, y=-1$ seems pretty nice so I'll use that. We will form the vector $\overrightarrow{AB}$ and find the vector projection of $\overrightarrow{AB}$ on $\vec{n}$. The distance to the plane will be the magnitude of that

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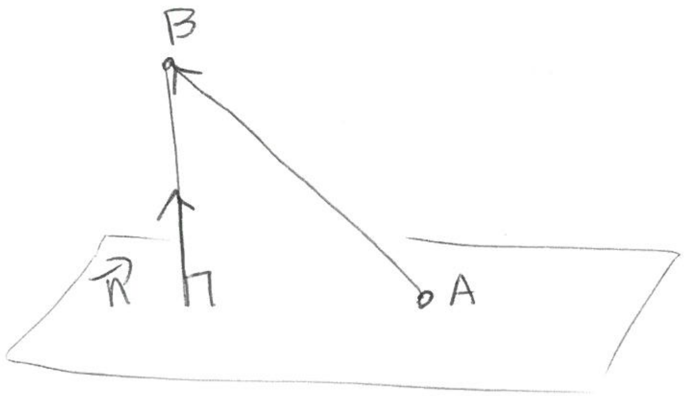
$A = (0,-1,0)$

$$\overrightarrow{AB} = \langle 1-0, 2+1, -4-0 \rangle = \langle 1, 3, -4 \rangle$$

From 1.3: $\text{proj}_{\vec{b}} \vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|} \dfrac{\vec{b}}{\|\vec{b}\|}$

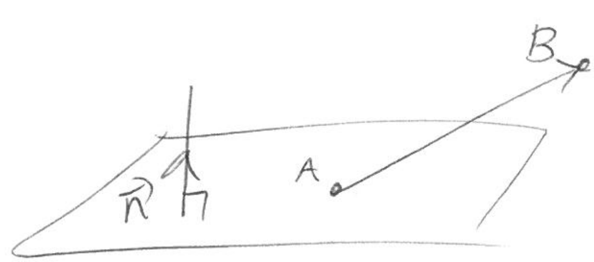
We want the length of that vector which is the scalar projection:

$$\text{comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|} = \frac{\overrightarrow{AB} \cdot \vec{n}}{\|\vec{n}\|} = \frac{\langle 1,3,-4 \rangle \cdot \langle 2,-1,3 \rangle}{\sqrt{2^2+1^2+3^2}}$$



$$= \frac{2-3-12}{\sqrt{14}} = \frac{-13}{\sqrt{14}}$$

Since this is negative our actual picture should look more like:

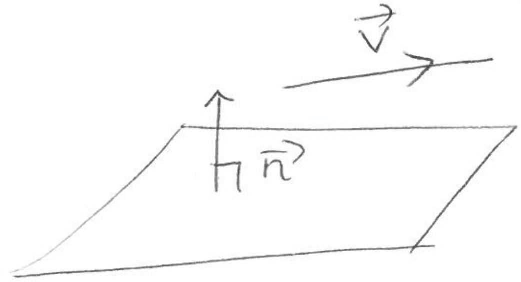


$$\boxed{d = \frac{13}{\sqrt{14}}}$$

$\boxed{\text{Ex 12}}$ Determine whether the line

$\vec{r}(t) = \langle 7-4t, 3+6t, 9+5t \rangle$ intersects or is parallel to the plane $4x+y+2z=17$

☆ Parallel:



Since $\vec{v}$ is in the direction of the line & $\vec{n}$ is $\perp$ to our plane, if they (the line & plane) are parallel $\vec{n}$ & $\vec{v}$ are $\perp$ so

$$\vec{n} \cdot \vec{v} = 0$$

$$\langle 4,1,2 \rangle \cdot \langle -4,6,5 \rangle = -16+6+10 = 0$$

Yes parallel

$\boxed{\text{Ex 13}}$ Same question as above

$\vec{r}(t) = \langle 3+2t, 6-5t, 2+3t \rangle$

$3x+2y-4z=1$

☆ Check to see if they're parallel

Is $\vec{n} \cdot \vec{v} = 0$?

$$\langle 3,2,-4 \rangle \cdot \langle 2,-5,3 \rangle = 6-10-12 \neq 0$$

not parallel

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This means our line & plane intersect so they have a point in common

$$\vec{r}(t) = \langle \underbrace{3+2t}_{x(t)}, \underbrace{6-5t}_{y(t)}, \underbrace{2+3t}_{z(t)} \rangle$$

$$3x + 2y - 4z = 1$$

$$3(3+2t) + 2(6-5t) - 4(2+3t) = 1$$

$$9 + 6t + 12 - 10t - 8 - 12t = 1$$

$$13 - 16t = 1$$

$$-16t = -12$$

$$t = \frac{12}{16} = \frac{3}{4}$$

$$\vec{r}\left(\tfrac{3}{4}\right) = \langle 3 + \tfrac{6}{4}, 6 - \tfrac{15}{4}, 2 + \tfrac{9}{4} \rangle$$

$$= \langle \tfrac{18}{4}, \tfrac{9}{4}, \tfrac{17}{4} \rangle$$

$$\boxed{pt = \left(\tfrac{18}{4}, \tfrac{9}{4}, \tfrac{17}{4}\right)}$$

Examples from Calculus books by Edwards/Penney & Briggs/Cochran/Gillett