Lagrange Multipliers

★ This topic is covered in my 242-040 & 242-050 courses. Feel free to read or ignore if you are in another one of my 242 sections.

To find the maximum & minimum values of f subject to the constraint $g=k$ (assuming these extreme values exist & $\nabla g \neq \vec{0}$ on the surface of $g=k$)

1. Find x, y, z so that $\boxed{\nabla f = \lambda \nabla g}$ & $\boxed{g=k}$

Lagrange multiplier
Sometimes we need to find this value

2. Find f at these values to find the max & min provided they exist

Examples from <u>Calculus</u> by James Stewart

Find the max & min values of f subject to the given constraint

Ex 1 $f(x,y) = x^2+y^2$ $\quad \underbrace{xy}_{g} = \underbrace{1}_{k}$

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle \quad \nabla g = \langle g_x, g_y \rangle = \langle y, x \rangle$$

$$\nabla f = \lambda \nabla g$$

$$\langle 2x, 2y \rangle = \lambda \langle y, x \rangle \longrightarrow \begin{array}{l} 2x = \lambda y \\ 2y = \lambda x \end{array}$$

☆ After finding your equations resulting from $\nabla f = \lambda \nabla g$ we need to solve for x & y. This is not necessarily easy & might require some clever thinking

$$2x = \lambda y \xrightarrow{\text{mult by } y} 2xy = \lambda y^2$$

$$2y = \lambda x \xrightarrow{\text{mult by } x} 2xy = \lambda x^2$$

$$xy = 1 \quad (g = k)$$

$$\lambda y^2 = \lambda x^2$$

$$\lambda(y^2 - x^2) = 0$$

$$\lambda = 0 \text{ or } y = \pm x$$

3

We know $2x = \lambda y$ so if $\lambda = 0$ then $2x = 0y = 0 \rightarrow x = 0$ which won't work for our constraint $xy = 1$

Therefore, $\lambda \neq 0$ which just leaves $y = x$ & $y = -x$

If $y = x$ then $xy = 1$ becomes $x^2 = 1$ or $x = 1$ or $x = -1$. Since we are starting with $y = x$, $x = 1 \rightarrow y = 1$ & $x = -1 \rightarrow y = -1$.

If $y = -x$ then $xy = 1$ becomes $-x^2 = 1$ which doesn't have real valued solutions

The 2 points we need to test are the ones we got from $y = x$:
$(1,1)$ & $(-1,-1)$

$f(1,1) = 1^2 + 1^2 = 2$

$f(-1,-1) = (-1)^2 + (-1)^2 = 2$

★ This tells us that for this given constraint we either don't have a max or we don't have a min

We can just think about it, for $xy=1$ we could have $x=1000$ & $y=\frac{1}{1000}$ then $f=x^2+y^2=1000^2+\frac{1}{1000^2}$ which is pretty clearly greater than 2.

2 = minimum value of f subject to our constraint, there is no max value.

$\boxed{\text{Ex 2}}$ $f(x,y)=x^2y$ $\quad \underbrace{x^2+2y^2}_{g}=6$

$$\nabla f=\langle \underbrace{2xy}_{f_x}, \underbrace{x^2}_{f_y} \rangle = \lambda \nabla g = \lambda \langle \underbrace{2x}_{g_x}, \underbrace{4y}_{g_y} \rangle$$

$2xy=\lambda 2x \longrightarrow 2xy-2x\lambda=0$

$2x(y-\lambda)=0 \rightarrow x=0$ or $y=\lambda$

$x^2=\lambda 4y$

If $x=0$, then $x^2=\lambda 4y$ becomes $0^2=\lambda 4y \rightarrow \lambda=0$ (y can't be zero when $x=0$ since we need $x^2+2y^2=6$)

5

$x=0 \rightarrow x^2+2y^2=6 \rightarrow 0^2+2y^2=6$

$y=\pm\sqrt{3}$

$y=\lambda \rightarrow x^2=\lambda 4y \rightarrow x^2=y4y$

$x^2=4y^2$

$x=\pm 2y$

If $x=2y$, then

$x^2+2y^2=6 \rightarrow (2y)^2+2y^2=6$

$4y^2+2y^2=6$

$6y^2=6$

$y^2=1$

$y=\pm 1$

Since $x=2y$ if $y=1$ $x=2$ & if $y=-1$ $x=-2$.

If $x=-2y$ then $x^2+2y^2=6 \rightarrow y^2=1$

$y=\pm 1$

If $y=1$, $x=-2$ & if $y=-1$, $x=2$.

6

$f(0,\sqrt{3}) = 0^2\sqrt{3} = 0$

$f(0,-\sqrt{3}) = 0^2(-\sqrt{3}) = 0$

$f(2,1) = 2^2\cdot 1 = 4$

$f(-2,-1) = (-2)^2(-1) = -4$

$f(-2,1) = (-2)^2(1) = 4$

$f(2,-1) = 2^2(-1) = -4$

$f = x^2y$

our candidates

Max value of $f = 4$

min value of $f = -4$

$\boxed{\text{Ex 3}}$ $f(x,y,z) = 8x - 4z$ $\quad \underbrace{x^2+10y^2+z^2}_{g} = 5$

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 8, 0, -4 \rangle$

$\nabla g = \langle g_x, g_y, g_z \rangle = \langle 2x, 20y, 2z \rangle$

$\nabla f = \lambda \nabla g$

$\langle 8, 0, -4 \rangle = \lambda \langle 2x, 20y, 2z \rangle$

$8 = 2\lambda x$

$0 = 20\lambda y \longrightarrow$ starting always with the easiest eqn either $\lambda = 0$ or $y = 0$

$-4 = 2\lambda z$

If $\lambda = 0$ $8 = 2\lambda x$ becomes $8 = 0$ which is obviously not true. Therefore $\lambda \neq 0$

7

Therefore $y=0$

$$8 = 2\lambda x \rightarrow x = \frac{4}{\lambda}$$

$$-4 = 2\lambda z \rightarrow z = \frac{-2}{\lambda}$$

we already know $\lambda \neq 0$. If we hadn't checked it previously we would need to.

$g=k$

$$x^2+10y^2+z^2=5$$

$$\left(\frac{4}{\lambda}\right)^2+10\cdot 0^2+\left(\frac{-2}{\lambda}\right)^2=5$$

$$\frac{16}{\lambda^2}+\frac{4}{\lambda^2}=5$$

$$16+4=5\lambda^2$$

$$20=5\lambda^2$$

$$\lambda=\pm 2$$

If $\lambda=2$, $x=\frac{4}{\lambda}=\frac{4}{2}=2$ & $z=\frac{-2}{\lambda}=\frac{-2}{2}=-1$

If $\lambda=-2$, $x=\frac{4}{\lambda}=-2$ & $z=\frac{-2}{\lambda}=1$

$$f=8x-4z$$

$f(2,0,-1)=8\cdot 2-4(-1)=20$ } max value

$f(-2,0,1)=8(-2)-4(1)=-20$ } min value

$\boxed{\text{Ex 4}}$ $f(x,y,z) = x+2y$ $\quad \underbrace{\underbrace{x+y+z=1}_{g} \quad \underbrace{y^2+z^2=4}_{h}}$

2 constraints!

I wouldn't ask a problem with 2 constraints on a test

Two constraints: To maximize/minimize f subject to $g=k$ & $h=c$ solve $\nabla f = \lambda \nabla g + \mu \nabla h$

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 1, 2, 0 \rangle$

$\nabla g = \langle g_x, g_y, g_z \rangle = \langle 1, 1, 1 \rangle$

$\nabla h = \langle h_x, h_y, h_z \rangle = \langle 0, 2y, 2z \rangle$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$\langle 1,2,0 \rangle = \lambda \langle 1,1,1 \rangle + \mu \langle 0, 2y, 2z \rangle$

$1 = \lambda 1 + \mu 0 = \lambda \longrightarrow \lambda = 1$

$2 = \lambda + \mu 2y$ since $\lambda = 1 \longrightarrow 2 = 1 + \mu 2y$

$0 = \lambda + \mu 2z \qquad \longrightarrow 0 = 1 + \mu 2z$

$$\left.\begin{array}{l} 2=1+\mu 2y \rightarrow 1=\mu 2y \\ 0=1+\mu 2z \rightarrow -1=\mu 2z \end{array}\right\} \text{add} \quad 0=2\mu y+2\mu z$$

$$0=2\mu(y+z)$$

either $\mu=0$ or $y=-z$

If $\mu=0$, $1=\mu 2y$ becomes $1=0$ which won't work. $\mu \neq 0$, so $y=-z$

$$y^2+z^2=4 \rightarrow (-z)^2+z^2=4 \rightarrow 2z^2=4$$

$$z^2=2$$

$$z=\pm\sqrt{2}$$

If $z=\sqrt{2}$ since $y=-z$ $y=-\sqrt{2}$

If $z=-\sqrt{2}$, $y=\sqrt{2}$.

$x+y+z=1$ so if $z=\sqrt{2}$, $y=-\sqrt{2}$ then

$$x+(-\sqrt{2})+\sqrt{2}=1 \rightarrow x=1$$

Likewise if $z=-\sqrt{2}$ & $y=\sqrt{2}$ $x+y+z=1$ becomes $x+\sqrt{2}-\sqrt{2}=1 \rightarrow x=1$

$f = x + 2y$

$f(1, -\sqrt{2}, \sqrt{2}) = 1 - 2\sqrt{2}$ } min

$f(1, \sqrt{2}, -\sqrt{2}) = 1 + 2\sqrt{2}$ } max

$\boxed{\text{Ex 5}}$ Find the extreme values of f on the region described by the inequality. $f(x,y) = e^{-xy}$ $x^2 + 4y^2 \leq 1$

☆ In the cases where we have $g \leq k$ also find & test the critical points of f.

$\nabla f = \langle f_x, f_y \rangle = \langle -ye^{-xy}, -xe^{-xy} \rangle$

$\nabla g = \langle g_x, g_y \rangle = \langle 2x, 8y \rangle$

$\nabla f = \lambda \nabla g$

$-ye^{-xy} = \lambda 2x \xrightarrow{\text{mult by } x} -xye^{-xy} = \lambda 2x^2$

$-xe^{-xy} = \lambda 8y \xrightarrow{\text{mult by } y} -xye^{-xy} = \lambda 8y^2$

so $\lambda 2x^2 = \lambda 8y^2$

$\lambda(2x^2 - 8y^2) = 0$ so $\lambda = 0$ or $2x^2 = 8y^2$

$x^2 = 4y^2$

$x = \pm 2y$

11

If $\lambda=0$, then $-ye^{-xy}=\lambda 2x=0 \rightarrow y=0$

& $-xe^{-xy}=\lambda 8y=0 \rightarrow x=0$

but we need $g=k$, $x^2+4y^2=1$

$0^2+4\cdot 0^2 \neq 1$

so $\lambda \neq 0$. ★ we need to approach this problem as $g=k$. we'll test $g<k$ by looking at f's critical points.

If $x=2y$, $x^2+4y^2=1$ becomes

$$(2y)^2+4y^2=1$$
$$4y^2+4y^2=1$$
$$8y^2=1$$
$$y=\pm\frac{1}{\sqrt{8}}$$

If $y=\frac{1}{\sqrt{8}}$, $x=2y \rightarrow x=2/\sqrt{8}$

If $y=-\frac{1}{\sqrt{8}}$, $x=2y \rightarrow x=-2/\sqrt{8}$

If $x=-2y$, $x^2+4y^2=1 \longrightarrow (-2y)^2+4y^2=1$

$y=\pm 1/\sqrt{8}$

If $y=\frac{1}{\sqrt{8}}$, $x=-2y \rightarrow x=-2/\sqrt{8}$

$y=-1/\sqrt{8}$, $x=-2y \rightarrow x=2/\sqrt{8}$

12

$f = e^{-xy}$

$f\left(\frac{2}{\sqrt{8}}, \frac{1}{\sqrt{8}}\right) = e^{-2/8} = e^{-1/4}$

$f\left(\frac{-2}{\sqrt{8}}, -1/\sqrt{8}\right) = e^{1/4}$

$f\left(\frac{-2}{\sqrt{8}}, \frac{1}{\sqrt{8}}\right) = e^{-1/4}$

$f\left(\frac{2}{\sqrt{8}}, \frac{-1}{\sqrt{8}}\right) = e^{-1/4}$

These would be the max & min values of f subject to $g = k$

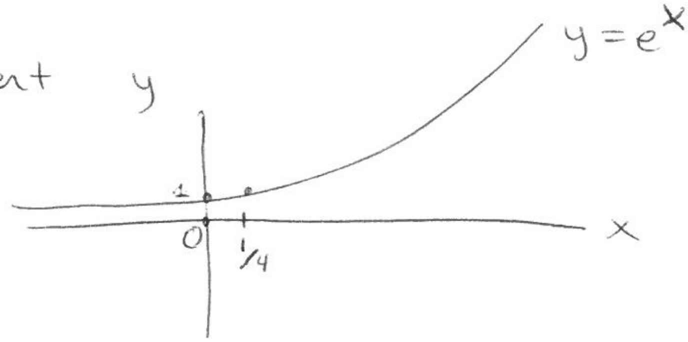
Critical points of f : $\nabla f = \vec{0}$

$\langle -ye^{-xy}, -xe^{-xy} \rangle = \langle 0, 0 \rangle \rightarrow y = 0, x = 0$

(This happens to be the values we found if $\lambda = 0$, but this won't always be the case).

$f(0,0) = e^0 = 1$

Note: $(0,0)$ meets the requirement $x^2 + 4y^2 \leq 1$



$e^{1/4}$ = max value

$e^{-1/4}$ = min value

$\boxed{\text{Ex 6}}$ Find the extreme values of f subject to the constraint

$f(x,y) = 2x^2 + 3y^2 - 4x - 5$ $\qquad \underbrace{x^2+y^2}_{g} \le 16$

$\nabla f = \langle f_x, f_y \rangle = \langle 4x-4, 6y \rangle$

$\nabla g = \langle g_x, g_y \rangle = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$

$\langle 4x-4, 6y \rangle = \lambda \langle 2x, 2y \rangle$

$4x - 4 = \lambda 2x$

$6y = \lambda 2y \rightarrow \quad 6y - \lambda 2y = 0$

$2y(3-\lambda) = 0$

$y = 0$ or $3 = \lambda$

If $y=0$ $\;x^2+y^2=16$ becomes $x^2+0^2=16$

so $x = \pm 4$

If $\lambda = 3$ then $4x-4 = \lambda 2x \longrightarrow 4x - 4 = 6x$

$-4 = 2x, \; x = -2$

$x^2+y^2=16 \rightarrow (-2)^2+y^2=16 \rightarrow y^2=12 \quad y = \pm\sqrt{12}$

14

$f = 2x^2 + 3y^2 - 4x - 5$

$$\left.\begin{aligned}
f(4,0) &= 32 + 0 - 16 - 5 = 11 \\
f(-4,0) &= 32 + 0 + 16 - 5 = 43 \\
f(-2,\sqrt{12}) &= 8 + 36 + 8 - 5 = 47 \\
f(-2,-\sqrt{12}) &= 8 + 36 + 8 - 5 = 47
\end{aligned}\right\}$$

if $g = k$
$x^2+y^2=16$
we would find our answer here

critical points of f: $\nabla f = \vec{0}$

$\langle 4x-4, 6y \rangle = \vec{0}$

$4x - 4 = 0 \rightarrow x = 1$

$6y = 0 \rightarrow y = 0$

Note: $(1,0)$ meets the requirement $x^2+y^2 \leq 16$

$f(1,0) = 2 + 0 - 4 - 5 = -7$

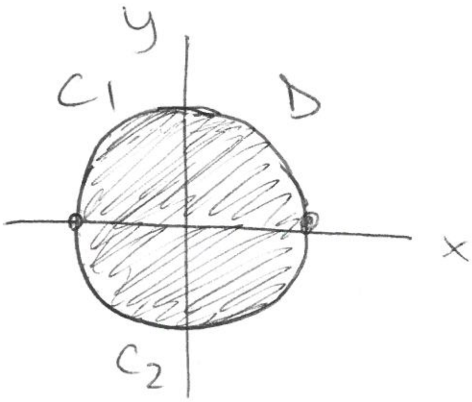
$$\boxed{\begin{aligned} &\text{Global min value} = -7 \\ &\text{Global max value} = 47 \end{aligned}}$$

★ Note: the last 2 examples could have been done with our global max/min values strategy.

15

Ex 6 Revisited Find the global max & min values of $f = 2x^2 + 3y^2 - 4x - 5$ on the region D given by $x^2 + y^2 \leq 16$

Without Lagrange multipliers, 1st we find critical points



$f_x = 4x - 4 = 0 \rightarrow x = 1$

$f_y = 6y = 0 \rightarrow y = 0$

Test the boundary $x^2 + y^2 = 16$

$y = \pm\sqrt{16 - x^2}$ (I solved for y since f is easier to plug into y because we have y^2)

C_1: $y = \sqrt{16 - x^2}$

$f(x, \sqrt{16-x^2}) = 2x^2 + 3(16 - x^2) - 4x - 5 = 43 - x^2 - 4x$

$f_x(x, \sqrt{16-x^2}) = -2x - 4 = 0 \quad x = -2$

$y = \sqrt{16-x^2} \rightarrow y = \sqrt{12}$

C_2: $y = -\sqrt{16-x^2}$

$f(x, -\sqrt{16-x^2}) = 43 - x^2 - 4x$

same as $C_1 \rightarrow x = -2 \quad y = -\sqrt{12}$

end points: points where the curves come together $(4,0)$ & $(-4,0)$

Candidates:

$f(1,0) = -7$

$f(-2, \sqrt{12}) = 47$

$f(-2, -\sqrt{12}) = 47$

$f(4,0) = 32 - 16 - 5 = 11$

$f(-4,0) = 32 + 16 - 5 = 43$

Global max value $= 47$

Global min value $= -7$

☆ To me, this seems easier. First Global max problem might be harder this way.

$\boxed{\text{Ex 7}}$ We can also use Lagrange multipliers to do more typical optimization problems.

Use Lagrange multipliers to find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$

distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$d = \sqrt{(x-4)^2 + (y-2)^2 + (z-0)^2}$

maximizing or minimizing the distance is equivalent to maximing/minimizing

$d^2 = (x-4)^2 + (y-2)^2 + z^2$

$f = (x-4)^2 + (y-2)^2 + z^2$

constraint:

$z^2 = x^2 + y^2$

so $\underbrace{x^2 + y^2 - z^2}_{g} = \underset{k}{\underset{\shortparallel}{0}}$

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2(x-4), 2(y-2), 2z \rangle$

$\nabla g = \langle g_x, g_y, g_z \rangle = \langle 2x, 2y, -2z \rangle$

$\nabla f = \lambda \nabla g$

17

$\cancel{2}(x-4) = \lambda \cancel{2}x$

$\cancel{2}(y-2) = \lambda \cancel{2}y$

$\cancel{2}z = -\lambda \cancel{2}z \rightarrow z + \lambda z = 0 \rightarrow z(1+\lambda) = 0$

so $z = 0$ or $\lambda = -1$

If $z=0$: $x^2+y^2-z^2=0$ (FYI: using $z^2-x^2-y^2=0$ will work too)

becomes $x^2+y^2=0 \rightarrow x=0, y=0$

If $\lambda=-1$: then $x-4 = \lambda x \rightarrow x-4 = -x \rightarrow 2x = 4$

$x = 2$

$y-2 = \lambda y \rightarrow y-2 = -y \rightarrow 2y = 2$

$y = 1$

then $x^2+y^2-z^2=0$ becomes $2^2+1^2-z^2=0$

$5 = z^2$

$z = \pm\sqrt{5}$

$f = (x-4)^2 + (y-2)^2 + z^2$

$f(0,0,0) = (-4)^2 + (-2)^2 + 0^2 = 20$

$f(2,1,\sqrt{5}) = (-2)^2 + (-1)^2 + (\sqrt{5})^2 = 10$

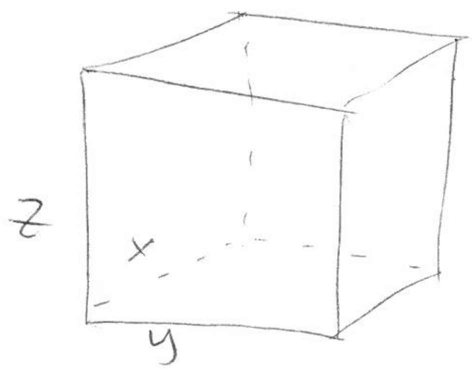
$f(2,1,-\sqrt{5}) = 2^2 + 1^2 + (-\sqrt{5})^2 = 10$

so shortest distance is $\sqrt{10}$ $(f = d^2)$

points closest are $\boxed{(2,1,\pm\sqrt{5})}$

Note: $(0,0,0)$ is not the furthest pt on the cone from $(4,2,0)$ as there is not a furthest point.

$\boxed{\text{Ex } 8}$ The base of an aquarium with given volume V is made of slate & the sides are made of glass. If slate costs five times as much per unit area as glass, find the dimensions of the aquarium that minimize cost.



$$V = xyz$$

$$C = 5xy + 1xz + 1xz + 1yz + 1yz$$

$$C = 5xy + 2xz + 2yz$$

f

$$C = 5xy + 2xz + 2yz$$

$$\underbrace{xyz}_{g} = V$$

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 5y + 2z, 5x + 2z, 2x + 2y \rangle$$

$$\nabla g = \langle yz, xz, xy \rangle$$

$$\nabla f = \lambda \nabla g$$

19

$5y+2z = \lambda(yz) \xrightarrow{\text{mult by } x} 5xy+2xz = \lambda xyz$

$5x+2z = \lambda xz \xrightarrow{\text{mult by } y} 5xy+2yz = \lambda xyz$

$2x+2y = \lambda xy$

$\cancel{5xy} + \cancel{2}xz = \cancel{5xy} + \cancel{2}yz$

$xz - yz = 0$

$(x-y)z = 0$

$x=y$ or $z=0$

$z=0$ would give us

$V=xyz=0$

so $z \neq 0$

If

$x=y$ $\quad 2x+2y=\lambda xy$ becomes

$4y=\lambda y^2 \rightarrow 0=\lambda y^2-4y=\lambda y(y-4)$

so $\lambda=0$ or $y=0$ or $y=4$

If $\lambda=0$ $\quad 5y+2z=\lambda yz=0$

So $5y=-2z$ $\quad x,y,z>0$

since they are distances &

we need $V\neq 0$, so $\lambda \neq 0$

If $y=0$, $V=xyz=0$, so $y\neq 0$

So $y=4$ since $x=y \rightarrow x=4$ then

$5x+2z=\lambda xz \rightarrow 20+2z=4\lambda z$

20

$$20 + 2z = 4\lambda z$$

$$2x + 2y = \lambda xy \xrightarrow[y=4]{x=4} 8 + 8 = \lambda 16 \rightarrow \lambda = 1$$

so $20 + 2z = 4\lambda z \rightarrow 20 + 2z = 4z$

$$20 = 2z \rightarrow z = 10$$

Dimensions that minimize cost! $x = 4$

$y = 4$

$z = 10$

There aren't dimensions that would maximize cost