MA 241 - 050 Test 3

1. (11 points) Find the general solution to $y'' + 5y' = 8$

2. (9 points) Show that if y_p is a particular solution to $ay'' + by' + cy = f(x)$, then ky_p is a particular solution to $ay'' + by' + cy = kf(x)$, where a, b, c, and k are constants.

3. (20 points) A spring with a mass of 1kg is stretched 2m by a 20N force. The frictional constant is $2\frac{\text{N sec}}{\text{m}}$ and there is an external force of $100e^{3t}$ N. If the spring is initially stretched 6m and with a velocity of 13m/s, find the position function of the mass.

4. (12 points) A population of rabbits grows at a rate proportional to its size. Suppose initially, we have 6 rabbits and after 30 days we have 24 rabbits. Find an equation for the number of rabbits after t days.

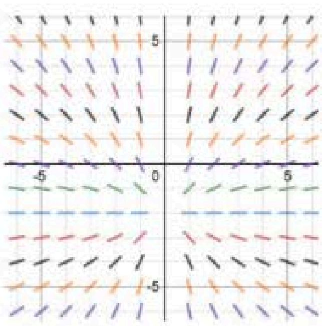
5. (20 points) a) Use Euler's method with a stepsize of 0.1 to estimate y(1.1) and y(1.2) where y(x) is a solution to the IVP $\frac{dy}{dx} = \frac{8x+12}{e^y}$; $y(1) = 0$

Clearly label your approximations. Note that y(1.2) doesn't need to be simplified.

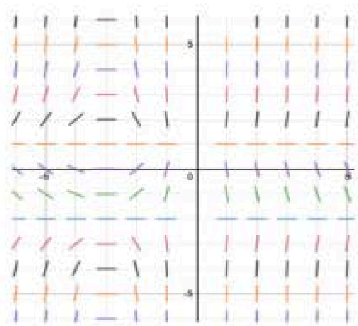
b) Find an explicit solution to the IVP $\frac{dy}{dx} = \frac{8x+12}{e^y}$; $y(1) = 0$

6. (20 points) A tank initially contains 100L of water in which 3 kg of salt has been dissolved. Pure water enters the tank at a rate of 4L/min. The solution is well-mixed and drains from the tank at a rate of 2L/min Find the amount of salt in the tank at time t.

7. (8 points) Find the equilibrium solutions of $\frac{dy}{dx} = \frac{(y+2)(y-1)(x+3)}{x}$ and match it to its slope field



A



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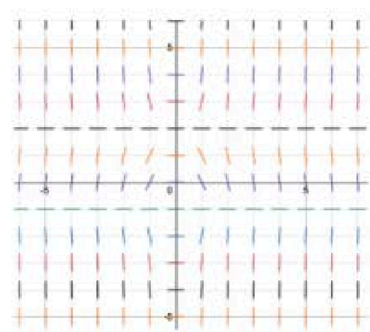

C

C2T3 Solutions

1. (11 pts)

$$r^2 + 5r = 0$$

$$r(r+5) = 0$$

$$y_c = c_1 + c_2 e^{-5t}$$

$$y_p = At$$

$$y'_p = A$$

$$y''_p = 0$$

$$0 + 5A = 8 \qquad A = 8/5$$

$$\boxed{y = c_1 + c_2 e^{-5t} + \frac{8}{5}t}$$

2. (9 pts)

$$aky_p'' + bky_p' + cky_p = k(ay_p'' + by_p' + cy_p)$$

$$= k\, f(x) \quad \checkmark$$

3. (20pts) $mx'' + bx' + kx = F_{ext}$

$$1x'' + 2x' + 10x = 100e^{3t}$$

$x(0) = 6$

$x'(0) = 13$

$F = kx$

$20 = k2 \quad k = 10$

$$r^2 + 2r + 10 = 0$$

$$r = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

$$x_c = e^{-t}\left[c_1 \cos 3t + c_2 \sin 3t\right]$$

$$x_p = Ae^{3t} \quad x'_p = 3Ae^{3t} \quad x''_p = 9Ae^{3t}$$

$$9Ae^{3t} + 2(3Ae^{3t}) + 10Ae^{3t} = 100e^{3t}$$

$$25A = 100 \quad A = 4 \quad x_p = 4e^{3t}$$

$$x = e^{-t}\left[c_1 \cos 3t + c_2 \sin 3t\right] + 4e^{3t}$$

$$x(0) = c_1 + 4 = 6 \quad c_1 = 2$$

$$x = e^{-t}\left[2\cos 3t + c_2 \sin 3t\right] + 4e^{3t}$$

$$x' = -e^{-t}\left[2\cos 3t + c_2 \sin 3t\right] + e^{-t}\left[-6\sin 3t + 3c_2 \cos 3t\right] + 12e^{3t}$$

$$x'(0) = 13 = -2 + 3c_2 + 12 \quad c_2 = 1$$

$$\boxed{x = e^{-t}\left[2\cos 3t + 1\sin 3t\right] + 4e^{3t}}$$

4. (12 pts)

$$y = y_0 e^{kt}$$

$$y = 6e^{kt}$$

$$y(30) = 24 = 6e^{30k}$$

$$4 = e^{30k}$$

$$\ln 4 = 30k$$

$$k = \frac{1}{30}\ln 4$$

$$\boxed{y = 6e^{\left(\frac{1}{30}\ln 4\right)t}}$$

5. (20 pts)

a) $y(1.1) \approx y_1 = y_0 + hf(x_0, y_0) = 0 + .1\left(\frac{8.1+12}{e^0}\right)$

$$= .1(20)$$

$$= 2$$

$y(1.2) \approx y_2 = y_1 + hf(x_1, y_1)$

$$= 2 + .1\left[\frac{8(1.1)+12}{e^2}\right]$$

b) $\int e^y dy = \int 8x + 12\, dx$

$$e^y = 4x^2 + 12x + C$$

$y = \ln(4x^2 + 12x + C)$ $\quad y(1) = 0$

$0 = \ln(4 + 12 + C) \rightarrow C = -15$ $\quad \boxed{y = \ln(4x^2 + 12x - 15)}$

6. (20 pts) $\boxed{100}$ 3kg

$$\frac{dx}{dt} = 8(4) - 2\left(\frac{x}{100+2t}\right)$$

$$= \frac{-x}{50+t}$$

$$\int \frac{dx}{x} = \int \frac{-1}{50+t}\, dt$$

$$\ln|x| = -\ln|50+t| + C$$

$$|x| = e^{-\ln|50+t| + C}$$

$$x = k e^{-\ln|50+t|} = \frac{k}{50+t}$$

$$x(0) = 3 = \frac{k}{50} \qquad k = 150$$

$$\boxed{x = \frac{150}{50+t}}$$

7. (8 pts) $y = -2,\ y = 1$

$\boxed{B}$