MA 242 Test 2 Version 2

1. (15 points) Find and classify any critical points of $f(x,y) = x^2y - 2x^2 - \frac{1}{4}y^2$ Fully justify your answers as we have done in class.

2. (16 points) Use $f(x,y,z) = \frac{x - z}{y + 1}$ to answer the following:

   a) Find the directional derivative of $f(x,y,z) = \frac{x - z}{y + 1}$ at $P(5,1,2)$ in the direction $Q(6,0,3)$

   b) Find the greatest rate of change of f at P

3. (20 points) Use $\mathbf{r}(t) = \langle 3\cos(t) + 1, 4\cos(t), 5\sin(t) + 2 \rangle$ to answer the following:

   a) Find the length of $\mathbf{r}(t)$ from $0 \leq t \leq 2\pi$

   b) Find a vector equation of the tangent line to $\mathbf{r}(t)$ when $t = 0$

4. (12 points) Use the chain rule to find $\frac{\partial(f \circ \mathbf{r})}{\partial u}$ at $(u,v) = (1,2)$ if $f(x,y) = \sqrt{x^2 + y^2}$ and $\mathbf{r}(u,v) = \langle \ln(u) + 3v, uv \rangle$

5. (20 points) Fully justify your work.

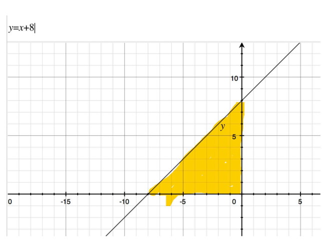
   a) Find the limit, if it exists, or show that the limit does not exist $\lim_{(x,y)\to(0,0)} \frac{\sqrt[3]{x}\, y^2}{2x + y^3}$

   b) Is $f(x,y) = \begin{cases} \frac{\sqrt[3]{x}\, y^2}{2x + y^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ continuous at (0,0)? Briefly explain why or why not.

   c) Find the limit, if it exists, or show that the limit does not exist $\lim_{(x,y)\to(3,3)} \frac{x - y}{x^2 - y^2}$

   d) b) Is $f(x,y) = \begin{cases} \frac{x - y}{x^2 - y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ continuous at (0,0)? Briefly explain why or why not.

6. (17 points) Find the global maximum and minimum values of $f(x,y) = xy + y^2$ on the region D shown below. Fully justify your answers as we have done in class. Make sure you clearly label your work!



C3 T2 Solutions

1. (15 points)

$f_x = 2xy - 4x = 0$ $\quad 2x(y-2) = 0$

$f_y = x^2 - \frac{1}{2}y = 0$ $\quad x = 0$ or $y = 2$

$x = 0 \rightarrow y = 0$

$y = 2 \rightarrow x = \pm 1$

Critical points $(0,0)$
$(1,2)$
$(-1,2)$

$f_{xx} = 2y - 4$

$f_{yy} = -\frac{1}{2}$

$f_{xy} = 2x$

$D = f_{xx} f_{yy} - [f_{xy}]^2 = (2y-4)(-\frac{1}{2}) - 4x^2$

$D(0,0) = (-4)(-\frac{1}{2}) = 2 > 0$

$f_{xx}(0,0) = -4 < 0$ $\quad (0,0) =$ local max

$D(1,2) = -4 < 0$ $\quad (1,2) =$ saddle pt

$D(-1,2) = -4 < 0$ $\quad (-1,2) =$ saddle pt

2. (16 pts)

a) $\nabla f = \left\langle \frac{1}{y+1}, -(x-z)(y+1)^{-2}, \frac{-1}{y+1} \right\rangle$

$\nabla f(5,1,2) = \left\langle \frac{1}{2}, \frac{-3}{4}, \frac{-1}{2} \right\rangle$

$\overrightarrow{PQ} = \langle 1, -1, 1 \rangle$

$\hat{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

$D_{\hat{u}} f = \nabla f \cdot \hat{u} = \left\langle \frac{1}{2}, \frac{-3}{4}, \frac{-1}{2} \right\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

$= \boxed{\frac{1}{2\sqrt{3}} + \frac{3}{4\sqrt{3}} - \frac{1}{2\sqrt{3}}}$

b) $\|\nabla f\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-3}{4}\right)^2 + \left(\frac{-1}{2}\right)^2}$

3. (20 pts)

a) $\vec{r}\,' = \langle -3\sin t, -4\sin t, 5\cos t \rangle$

$\|\vec{r}\,'\| = \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t}$

$25\sin^2 t$

$= \sqrt{25} = 5$

$L = \int_0^{2\pi} 5\,dt = \boxed{10\pi}$

b) $\vec{r}\,'(0) = \langle 0, 0, 5 \rangle$

$\vec{r}(0) = \langle 4, 4, 2 \rangle$

$\vec{r}_{\text{Tangent}} = \langle 4, 4, 2 \rangle + \langle 0, 0, 5 \rangle t$

4. (12 points)

$$\frac{\partial(f\circ\vec{r})}{\partial u} = \nabla f(\vec{r}) \cdot \frac{\partial \vec{r}}{\partial u}$$

$$\nabla f = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

$$\frac{\partial \vec{r}}{\partial u} = \left\langle \frac{1}{u}, v \right\rangle$$

$$\frac{\partial \vec{r}}{\partial u}(1,2) = \langle 1, 2 \rangle$$

$$\left.\begin{matrix} u=1 \\ v=2 \end{matrix}\right\} \begin{matrix} x = \ln 1 + 6 = 6 \\ y = 2 \end{matrix}$$

$$\frac{\partial(f\circ\vec{r})}{\partial u} = \left\langle \frac{6}{\sqrt{40}}, \frac{2}{\sqrt{40}} \right\rangle \cdot \langle 1, 2 \rangle$$

$$= \frac{6}{\sqrt{40}} + \frac{4}{\sqrt{40}} = \boxed{\frac{10}{\sqrt{40}}}$$

5. (20 pts)

a) $x=0$ $\lim\limits_{(0,y)\to(0,0)} \frac{0}{y^3} = 0$

$x=y^3$ $\lim\limits_{(y^3,y)\to(0,0)} \frac{\sqrt[3]{y^3}\, y^2}{2y^3+y^3} = \lim\limits_{y\to 0} \frac{y^3}{3y^3} = \frac{1}{3}$

$0 \neq \frac{1}{3}$ limit DNE

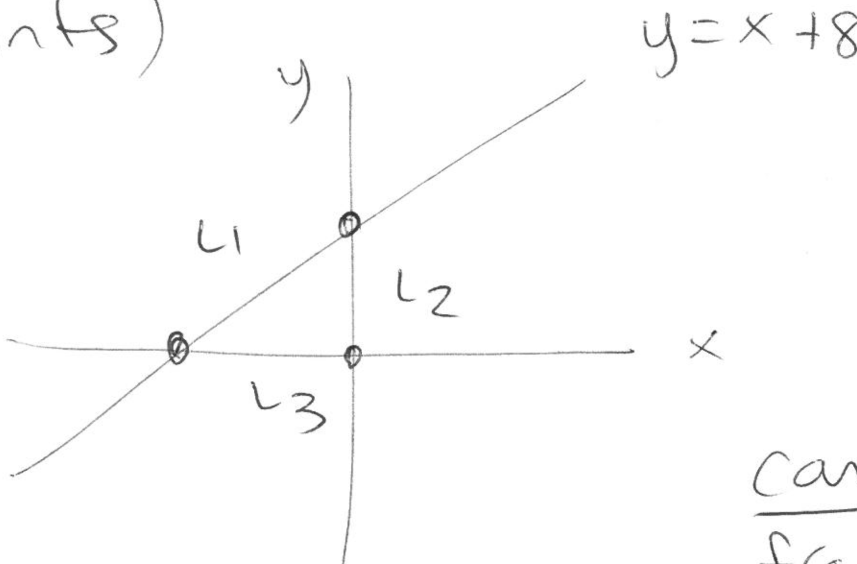
b) No, limit DNE

c) $\lim\limits_{(x,y)\to(3,3)} \frac{\cancel{(x-y)}}{\cancel{(x-y)}(x+y)} = \boxed{\frac{1}{6}}$

d) $\lim\limits_{(x,y)\to(0,0)} \frac{x-y}{x^2-y^2} = \frac{1}{0+0}$ DNE

No limit DNE

6. (17 points)



Candidates

$f(0,0) = 0$

$f(-6,2) = -12+4 = -8$

$f(0,0) = 0$

$f(0,0) = 0$

$f(-8,0) = 0$

$f(0,8) = 64$

$f = xy + y^2$

$f_x = y = 0 \quad y = 0$

$f_y = x + 2y = 0 \quad x = 0$

$L_1: f(x, x+8) = x(x+8) + (x+8)^2$

$= x^2 + 8x + x^2 + 16x + 64$

$f_x(x, x+8) = 2x + 8 + 2x + 16$

$= 4x + 24 \quad x = -6$

$L_2: x = 0 \quad f(0,y) = y^2$

$f_y(0,y) = 2y = 0$

$y = 0$

$L_3: y = 0 \quad f(x,0) = 0$

$$\boxed{\begin{array}{l}\text{Global max value} = 64 \\ \text{Global min} = -8\end{array}}$$