1. (17 points) Use the **method of undetermined coefficients** to solve the initial value problem:

$y'' + 2y' + 2y = 12t^2 + 18t \quad y(0) = 0,\ y'(0) = 0$

2. (14 points) Use the **method of variation of parameters** to find the general solution to

$y'' - 14y' + 49y = 6e^{7t}$

3. (18 points) An object with a 1 kg mass stretches a spring 5 m before coming to rest at equilibrium. The spring has a damping constant of 3Nsec/m and the spring is compressed 1 meter from its equilibrium position and released.

a) Use 10 m/sec^2 for gravity and formulate the IVP to find the position of the mass at time t.

b) What kind of damping is this?

c) If an external force of 3 N is added at t = 3 sec and taken away at t = 12 sec, adjust the differential equation you set up in part a) to reflect this. Just set up the differential equation; do not solve it.

4. (13 points) Use the definition of the Laplace transform to find the Laplace of $f(t) = \begin{cases} 0 & \text{if } t < 1 \\ 6e^{2t} & \text{if } t \geq 1 \end{cases}$

Determine its domain.

Laplace table for problems 5 and 6

$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$ $\qquad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$ $\qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $\qquad \mathcal{L}\{1\} = \frac{1}{s}$

$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$ $\qquad \mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$ $\qquad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $\qquad \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$

$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$ $\qquad \mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$ $\qquad \mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$

$\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$ $\qquad \mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$ $\qquad \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n(F(s))}{ds^n}$

5. (19 points) Find the inverse Laplace transform of the following:

a) $\dfrac{7s^2+26}{s(s^2-4s+13)}$

b) $e^{-4s}\left(\dfrac{18s}{s^2+4}\right)$

6. (19 points) Use the **method of Laplace transforms** to solve the following initial value problem,

$y'' - 6y' + 9y = 18\ ;\ y(0) = 0,\ y'(0) = 6$

1. (17 points) $r^2 + 2r + 2 = 0$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$y_c = e^{-t}\left[C_1 \cos t + C_2 \sin t\right]$$

$$y_p = At^2 + Bt + C$$
$$y_p' = 2At + B$$
$$y_p'' = 2A$$

$$2A + 2(2At + B) + 2(At^2 + Bt + C) = 12t^2 + 18t$$

$2A = 12 \qquad A = 6$

$4A + 2B = 18 \qquad 24 + 2B = 18 \qquad B = -3$

$2A + 2B + 2C = 0 \qquad 12 - 6 + 2C = 0$

$C = -3$

$$y = e^{-t}\left[C_1 \cos t + C_2 \sin t\right] + 6t^2 - 3t - 3$$

$y(0) = 0 = C_1 - 3 \qquad C_1 = 3$

$$y = e^{-t}\left[3\cos t + C_2 \sin t\right] + 6t^2 - 3t - 3$$

$$y' = -e^{-t}\left[3\cos t + C_2 \sin t\right] + e^{-t}\left[3(-\sin t) + C_2 \cos t\right]$$
$$+ 12t - 3$$

$y'(0) = 0 = -3 + C_2 - 3 \qquad C_2 = 6$

$$y = e^{-t}\left[3\cos t + 6\sin t\right] + 6t^2 - 3t - 3$$

2. (14 pts) $r^2 - 14r + 49 = 0$

$$(r-7)^2 = 0$$

$$y_c = c_1 e^{7t} + c_2 t e^{7t}$$

$$y_p = v_1 e^{7t} + v_2 t e^{7t}$$

$$v_1' e^{7t} + v_2' t e^{7t} = 0$$

$$v_1' = -v_2' t$$

$$v_1' 7e^{7t} + v_2'(e^{7t} + 7te^{7t}) = 6e^{7t}$$

$$\cancel{-v_2' t\, 7e^{7t}} + v_2'(e^{7t} + \cancel{7te^{7t}}) = 6e^{7t}$$

$$v_2' = 6 \qquad v_2 = 6t$$

$$v_1' = -6t \qquad v_1 = -3t^2$$

$$y = c_1 e^{7t} + c_2 t e^{7t} - 3t^2 e^{7t} + 6t^2 e^{7t}$$

3. (~~20~~ 18 points)

$F = kx$

$10 = k(5)$

a) $\boxed{x'' + 3x' + 2x = 0 \qquad x(0) = -1 \quad x'(0) = 0}$

~~$b^2 - 4ac$~~ $b^2 - 4mk = 9 - 8 = 1 > 0$

b) over damping

c) $\boxed{x'' + 3x' + 2x = 3u(t-3) - 3u(t-12)}$

d) $s^2\mathcal{L}\{x\} - s\overset{(-1)}{\cancel{x(0)}} - \overset{0}{\cancel{x'(0)}} + 3[s\mathcal{L}\{x\} - \overset{(-1)}{\cancel{x(0)}}] + 2\mathcal{L}\{x\}$

$$= \frac{3e^{-3s}}{s} - \frac{3e^{-12s}}{s}$$

4. (13 points)

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f\,dt$$

$$= \int_1^\infty e^{-st} 6e^{2t}\,dt$$

$$= \lim_{n\to\infty} \int_1^n 6e^{(2-s)t}\,dt$$

$$\overset{\lim_{n\to\infty}}{=} \frac{6}{2-s} e^{(2-s)t} \Big|_1^n = \lim_{n\to\infty} \frac{6}{2-s} e^{(2-s)n} - \frac{6}{2-s} e^{2-s}$$

$$= \boxed{\frac{-6}{2-s} e^{2-s}} \quad \boxed{s > 2}$$

5. (19 pts)

a) $\frac{A}{s} + \frac{B(s-2)+3C}{(s-2)^2+9}$

$s^2 - 4s + 4 + 9$ ✓

$As^2 - \underline{4As} + 13A + Bs^2 \underline{-2Bs} + \underline{3Cs} = 7s^2 + 26$

$A + B = 7$ $\qquad B = 5$

$-4A - 2B + 3C = 0 \qquad -8 - 10 + 3C = 0$

$13A = 26 \quad A = 2 \qquad C = 6$

$\frac{2}{s} + \frac{5(s-2) + 3(6)}{(s-2)^2+9}$

$$\boxed{2 + 5\cos 3t\, e^{2t} + 6\sin 3t\, e^{2t}}$$

5 b) $\frac{18s}{s^2+4} \to 18\cos 2t$

$\mathcal{L}^{-1}\{e^{-4s}F(s)\} = f(t-4)u(t-4)$

$= \boxed{18\cos(2(t-4))u(t-4)}$

6. (17 pts)

$s^2\mathcal{L}\{y\} - \cancel{sy(0)} - \cancel{y'(0)}^{6} - 6[s\mathcal{L}\{y\} - \cancel{y(0)}] + 9\mathcal{L}\{y\}$

$= \frac{18}{s}$

$(s^2 - 6s + 9)L = \frac{18}{s} + 6$

$L = \frac{\frac{18}{s} + 6}{s^2-6s+9} \cdot \frac{s}{s} = \frac{18+6s}{s(s-3)^2}$

$= \frac{A}{s} + \frac{B}{s-3} + \frac{C}{(s-3)^2}$

$A(s^2-6s+9) + Bs^2 - 3Bs + Cs = 18 + 6s$

$A + B = 0$

$-6A - 3B + C = 6$ $\qquad -12 + 6 + C = 6$

$9A = 18 \quad A = 2 \quad B = -2 \qquad C = 12$

$\boxed{y = 2 - 2e^{3t} + 12te^{3t}}$