MA 341 Test 3 Version 1 Show all of your work!

1. (18 points) a) Write the following system of equations in the form $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$

$$x_1' = 3x_1 - 6x_2 + 4e^{2t}$$
$$x_2' = x_2 - 1$$

b) Use the method of undetermined coefficients to find its particular solution

Hint: Don't find $\vec{\mathbf{x}}_c$!

2. (27 points) a) Find the general solution to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ where $A = \begin{bmatrix} 3 & 3 & 4 \\ 1 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ has the characteristic polynomial $(r-2)^2(r-6) = 0$

b) Augment A with the identity matrix and use row operations so that the first column of A becomes $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ Note: You are just doing the first couple of steps in finding A^{-1}

3. (16 points) Find the general solution of $\vec{\mathbf{x}}' = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix}\vec{\mathbf{x}}$

Hint: If $r = \alpha + \beta i$ with $\mathbf{u} = \mathbf{a} + \mathbf{b}i$, two linearly independent solutions are

$e^{\alpha t}\cos(\beta t)\mathbf{a} - e^{\alpha t}\sin(\beta t)\mathbf{b}$ and $e^{\alpha t}\sin(\beta t)\mathbf{a} + e^{\alpha t}\cos(\beta t)\mathbf{b}$

4. (23 points) Use $\vec{\mathbf{x}}_c = c_1 e^{8t}\begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ to find the particular solution to $\vec{\mathbf{x}}' = \begin{bmatrix} 3 & 1 \\ 10 & 6 \end{bmatrix}\vec{\mathbf{x}} + \begin{bmatrix} e^t \\ 12e^t \end{bmatrix}$ using the method of variation of parameters.

Hint: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

5. (16 points) Tank A initially holds 6L of brine solution containing 0.2kg of salt, while tank B initially contains 50L of pure water. The tanks are connected by pipes so that the liquid in tank A flows into tank B at a rate of 3L/min and from tank B into tank A at a rate of 1L/min. A solution containing 0.1 kg/L of salt is poured into tank A at a rate of 8 L/min while a solution containing 0.3 kg/L of salt is poured into tank B at a rate of 5 L/min. Both tanks are kept thoroughly mixed. The contents of tank A flow out of a drain at the bottom of tank A at a rate of 6 L/min while the contents of tank B flow out of a drain out of the bottom of tank B at a rate of 7 L/min. If $x_1(t)$ is the amount of salt in tank A and $x_2(t)$ is the amount of salt in tank B, find $\mathbf{A}$, $\vec{\mathbf{f}}$, and $\vec{\mathbf{x}}(\mathbf{0})$ so that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \mathbf{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \vec{\mathbf{f}}(t)$

MA 341 T3 Solutions

1. (18 points)

a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 3 & -6 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4e^{2t} \\ -1 \end{bmatrix}$

b) $\vec{x_p} = \vec{a}e^{2t} + \vec{b}$

$\vec{x_p}' = 2\vec{a}e^{2t}$

$$2\vec{a}e^{2t} = \begin{bmatrix} 3 & -6 \\ 0 & 1 \end{bmatrix}\vec{a}e^{2t} + \begin{bmatrix} 3 & -6 \\ 0 & 1 \end{bmatrix}\vec{b} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}e^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2a_1 \\ 2a_2 \end{bmatrix} = \begin{bmatrix} 3a_1 - 6a_2 \\ a_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$a_2 = 0$

$2a_1 = 3a_1 + 4$

$-a_1 = 4 \qquad a_1 = -4$

$\vec{0} = \begin{bmatrix} 3b_1 - 6b_2 \\ b_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad b_2 = 1$

$3b_1 - 6 = 0 \qquad b_2 = 2$

$$\vec{x_p} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}e^{2t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -4e^{2t} + 1 \\ 2 \end{bmatrix}}$$

2. (27 points)

a) $(A-2I)\vec{u}=\vec{0}$

$$\begin{bmatrix} 1 & 3 & 4 \\ 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} = \vec{0}$$

$u_a = -3u_b - 4u_c$

$$\vec{u}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

$(A-6I)\vec{u}=\vec{0}$

$$\begin{bmatrix} -3 & 3 & 4 \\ 1 & -1 & 4 \\ 0 & 0 & -4 \end{bmatrix}\vec{u} = \vec{0}$$

$$\vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$u_c = 0 \quad u_a = u_b$

$$\vec{x} = c_1 e^{2t}\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{6t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

b)

$$\left[\begin{array}{ccc|ccc} 3 & 3 & 4 & 1 & 0 & 0 \\ 1 & 5 & 4 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - 3R_2}$$

$$\left[\begin{array}{ccc|ccc} 0 & -12 & -8 & 1 & -3 & 0 \\ 1 & 5 & 4 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 4 & 0 & 1 & 0 \\ 0 & -12 & -8 & 1 & -3 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

3. (16 points)

$$\begin{vmatrix} 2-r & 1 \\ -4 & 2-r \end{vmatrix} = (2-r)^2 + 4 = 0$$

$$4 - 4r + r^2 + 4 = 0$$

$$r^2 - 4r + 8 = 0$$

$$r = \frac{4 \pm \sqrt{16-32}}{2} = \frac{4 \pm \sqrt{-16}}{2} = \frac{4 \pm 4i}{2}$$

$$= 2 \pm 2i$$

$$(A - (2+2i)I)\vec{u} = \vec{0}$$

$$\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix} \vec{u} = \vec{0}$$

$$-4u_a - 2i u_b = 0$$

$$u_a = \frac{-2i u_b}{4} = -\frac{1}{2} i u_b$$

$$\vec{u} = \begin{bmatrix} -i \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$$

$$\vec{x} = c_1 \left(e^{2t} \cos 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix} - e^{2t} \sin 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

$$+ c_2 \left(e^{2t} \sin 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix} + e^{2t} \cos 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

4. (23 pts)

$$\underline{X} = \begin{bmatrix} e^{8t} & -e^{t} \\ 5e^{8t} & 2e^{t} \end{bmatrix}$$

$$\underline{X}^{-1} = \frac{1}{2e^{9t}+5e^{9t}} \begin{bmatrix} 2e^{t} & e^{t} \\ -5e^{8t} & e^{8t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{7}e^{-8t} & \frac{1}{7}e^{-8t} \\ -\frac{5}{7}e^{-t} & \frac{1}{7}e^{-t} \end{bmatrix}$$

$$\vec{x}_p = \underline{X} \int \underline{X}^{-1} \vec{f}\, dt$$

$$= \underline{X} \int \begin{bmatrix} \frac{2}{7}e^{-8t} & \frac{1}{7}e^{-8t} \\ -\frac{5}{7}e^{-t} & \frac{1}{7}e^{-t} \end{bmatrix} \begin{bmatrix} e^{t} \\ 12e^{t} \end{bmatrix} dt$$

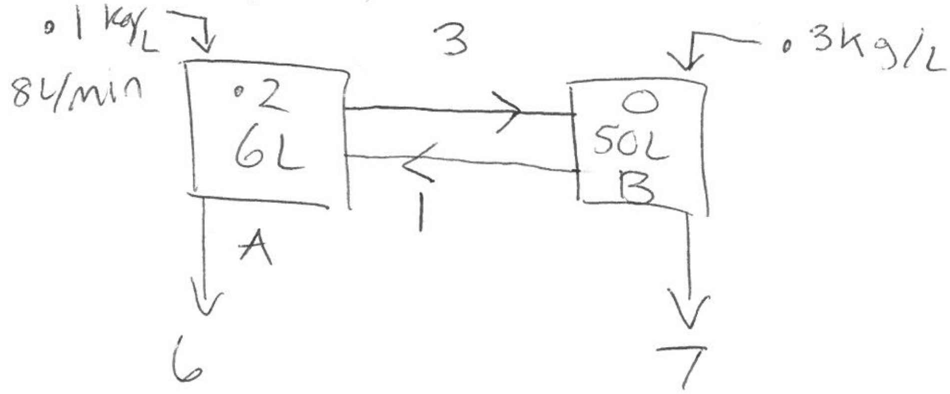
$$= \underline{X} \int \begin{bmatrix} \frac{2}{7}e^{-7t} + \frac{12}{7}e^{-7t} \\ -\frac{5}{7} + \frac{12}{7} \end{bmatrix} dt$$

$$= \underline{X} \int \begin{bmatrix} 2e^{-7t} \\ 1 \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{8t} & -e^{t} \\ 5e^{8t} & 2e^{t} \end{bmatrix} \begin{bmatrix} -2/7\, e^{-7t} \\ t \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{7}e^{t} - te^{t} \\ -\frac{10}{7}e^{t} + 2te^{t} \end{bmatrix}$$

5. (16 pts)



$$\frac{dx_1}{dt} = (8)(.1) + 1\left(\frac{x_2}{50}\right) - \frac{x_1}{6}(9)$$

$$\frac{dx_2}{dt} = 5(.3) + 3\,\frac{x_1}{6} - \frac{x_2}{50}(8)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -\frac{9}{6} & \frac{1}{50} \\ \frac{1}{2} & -\frac{8}{50} \end{bmatrix} + \begin{bmatrix} .8 \\ 1.5 \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} .2 \\ 0 \end{bmatrix}$$