MA 341 T3 Solutions

1. (25 points)

$$X = \begin{bmatrix} -2e^{2t} & e^{5t} \\ e^{2t} & e^{5t} \end{bmatrix}$$

$$X^{-1} = \frac{1}{-2e^{7t} - e^{7t}} \begin{bmatrix} e^{5t} & -e^{5t} \\ -e^{2t} & -2e^{2t} \end{bmatrix}$$

$$-\tfrac{1}{3}e^{-7t}$$

$$X^{-1} = \begin{bmatrix} -\frac{1}{3}e^{-2t} & \frac{1}{3}e^{-2t} \\ \frac{1}{3}e^{-5t} & \frac{2}{3}e^{-5t} \end{bmatrix}$$

$$\vec{x_p} = X \int X^{-1} \vec{f}\, dt$$

$$= X \int \begin{bmatrix} -\frac{1}{3}e^{-2t} & \frac{1}{3}e^{-2t} \\ \frac{1}{3}e^{-5t} & \frac{2}{3}e^{-5t} \end{bmatrix} \begin{bmatrix} 3e^{2t} \\ -6e^{2t} \end{bmatrix}$$

$$= X \int \begin{bmatrix} -1 - 2 \\ 1e^{-3t} - 4e^{-3t} \end{bmatrix} dt$$

$$= X \int \begin{bmatrix} -3 \\ -3e^{-3t} \end{bmatrix} dt$$

$$= \begin{bmatrix} -2e^{2t} & e^{5t} \\ e^{2t} & e^{5t} \end{bmatrix} \begin{bmatrix} -3t \\ e^{-3t} \end{bmatrix}$$

$$\boxed{\vec{x_p} = \begin{bmatrix} 6te^{2t} + e^{2t} \\ -3te^{2t} + e^{2t} \end{bmatrix}}$$

2. (16 points)

$$|A - rI| = \begin{vmatrix} 4-r & -3 & -2 \\ 0 & 7-r & 2 \\ 0 & 0 & 4-r \end{vmatrix}$$

$$= (4-r)^2(7-r) = 0$$

$$r_1 = r_2 = 4 \quad r_3 = 7$$

$$(A - 4I)\vec{u} = \vec{0}$$

$$\begin{bmatrix} 0 & -3 & -2 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} = \vec{0}$$

$$3u_b + 2u_c = 0$$

$$u_b = -\frac{2}{3}u_c$$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

$$(A - 7I)\vec{u}_3 = \vec{0}$$

$$\begin{bmatrix} -3 & -3 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix}$$

$$-3u_a - 3u_b = 0$$

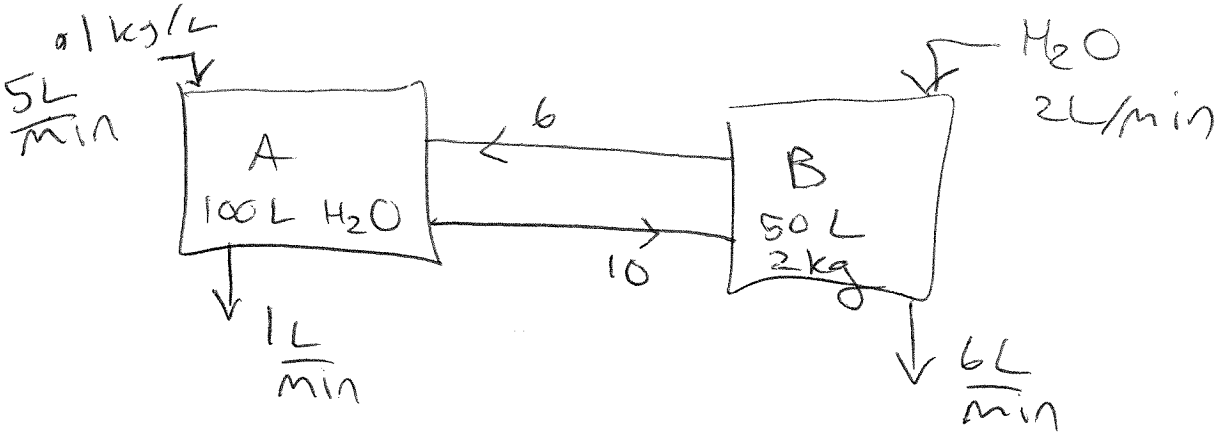
$$u_a = -u_b$$

$$-3u_c = 0$$

$$\vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{x} = c_1 e^{4t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} + c_3 e^{7t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

3. (15 pts)



$$\frac{dx_1}{dt} = 5(.1) + 6\left(\frac{x_2}{50}\right) - 11\left(\frac{x_1}{100}\right)$$

$$\frac{dx_2}{dt} = 2(0) + 10\left(\frac{x_1}{100}\right) - 12\left(\frac{x_2}{50}\right)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -11/100 & \frac{6}{50} \\ \frac{10}{100} & \frac{-12}{50} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} .5 \\ 0 \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

4. (15 pts)

$$|A - rI| = \begin{vmatrix} 4-r & 1 \\ -5 & -r \end{vmatrix} = (4-r)(-r) + 5$$

$$r^2 - 4r + 5 = 0$$

$$r = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$$(A - (2+i)I)\vec{u} = \vec{0}$$

$$\begin{bmatrix} 2-i & 1 \\ -5 & -2-i \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix} = \vec{0}$$

$$(2-i)u_a + 1u_b = 0 \qquad u_b = -(2-i)u_a$$

$$\vec{u} = \begin{bmatrix} 1 \\ -2+i \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i$$

$$\vec{x} = c_1 e^{2t}\left[\cos t \begin{bmatrix} 1 \\ -2 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] + c_2 e^{2t}\left[\cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right]$$

or

$$-5u_a + (-2-i)u_b = 0 \qquad u_a = \frac{(2+i)u_b}{-5}$$

$$\vec{u} = \begin{bmatrix} 2+i \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$$

$$\vec{x} = c_1 e^{2t}\left[\cos t \begin{bmatrix} 2 \\ -5 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right] + c_2 e^{2t}\left[\cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 2 \\ -5 \end{bmatrix}\right]$$

5. (17 points)

$$\vec{x_p} = \vec{a}t + \vec{b}$$

$$\vec{x_p}' = \vec{a}$$

$$\vec{x_p}' = \begin{bmatrix} 4 & 1 \\ -5 & 0 \end{bmatrix}(\vec{a}t + \vec{b}) + \begin{bmatrix} -4t + 3 \\ 5t \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} 4 & 1 \\ -5 & 0 \end{bmatrix}\vec{a}t + \begin{bmatrix} 4 & 1 \\ -5 & 0 \end{bmatrix}\vec{b} + \begin{bmatrix} -4 \\ 5 \end{bmatrix}t + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\vec{0} = \begin{bmatrix} 4 & 1 \\ -5 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 4a_1 + a_2 \\ -5a_1 \end{bmatrix} \qquad a_1 = 1 \rightarrow a_2 = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -5 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4b_1 + b_2 \\ -5b_1 \end{bmatrix} \qquad b_1 = 0 \quad b_2 = -2$$

$$\vec{x_p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}t + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} t \\ -2 \end{bmatrix}$$

6. (12 points)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$-R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 - 3R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -6 & 3 & 1 \end{array}\right]$$

$R_1 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -1 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -6 & 3 & 1 \end{array}\right]$$

A^{-1}